Learning through Transient Matching in Congested Markets[*]

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Abstract

I introduce a framework for studying transient matching in decentralized markets where workers learn about their preferences through their experiences. Limits on the number of available positions force workers to compete over matches. Each capacity-constrained firm employs workers whose match value exceeds a threshold. Since employment offers both payoff and information benefits, workers effectively face a multi-armed bandit problem. To them each firm acts as a bandit where the probability of "success" at the firm is driven by market competition. In such markets, aggregate demand for firms satisfies the gross substitutes condition which ensures equilibrium existence. The resulting search patterns match a variety of stylized facts from labor market data. High-quality workers search less and tenure increases with age. In general, equilibria are inefficient because competition depresses the level of search. Natural interventions designed to improve efficiency are effective in uncongested markets, but can fail when congestion is several. From a market design perspective, the utilization of headhunters has differential effects depending on workers' quality, conclusively improving both outcomes for low-quality workers and overall efficiency. Reducing congestion through unemployment benefits, can depress search and may ultimately reduce match efficiency.

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1 Introduction

1.1 Overview

In many markets, participants learn about potential matches via pairing. In labor markets, employees often learn about a company's culture on the job; in residency markets, aspiring doctors learn about their preferred specialties through apprenticeships; and in marriage markets, individuals learn about prospective spouses through dating. Such markets are often congested: Firms hire a limited number of employees; hospitals have federal funding limits capping the number of residents they can hire; and many relationships are monogamous. Congestion limits the ability of agents to learn: a worker who is not hired cannot learn her match value with a firm. How do agents strategically search in congested matching markets? When agents learn through matching, who ultimately is matched with whom? How do common interventions—hiring intermediaries or increasing unemployment benefits—change the quality of matches?

This paper develops a novel model of learning through matching when there is a limited number of positions available. I extend techniques from the operations literature on multi-armed bandits to determine workers' equilibrium search patterns, when the rewards from search are endogenous. I characterize the set of equilibria in congested markets with transient matching. In general, equilibria are inefficient due to competition depressing the level of search. In line with empirical work on tenure, in equilibrium, workers with higher unanticipated match values, as well as older workers, search less (Gorry 2016). I consider the impact of two common policy interventions: introducing informed intermediaries—headhunters—and increasing unemployment benefits. Both interventions unambiguously improve welfare when the market is not congested. I show that revealing information about a firm's match values through an intermediary still improves *total* equilibrium welfare, though benefits are unequally distributed among workers. Increasing unemployment benefits intensifies competition—which can be detrimental to welfare—when there is a commonly known top firm. In contrast, when markets are uncongested, unemployment benefits always improve equilibrium welfare.

In the model, a continuum of workers repeatedly search for jobs at a finite number of firms. Workers differ in their observable characteristics, but do not know their fit at a given firm until they are hired by that firm.[1] Each period, every worker applies to a single firm. Firms interview their set of applicants and hire the workers with the best fit subject to capacity constraints. Hired workers learn about the quality of their match, while rejected workers only learn of their rejections.[2] Workers and firms split the surplus from matching. At the end of the period,

[1]For instance, a young graduate of computer science would know her grades and the school she attended, but might not know that a position at Google would feature her best fit.

[2]Results in the empirical literature on search motivate the modelling choice that workers only learn upon being matched (Menzio, Telyukova, and Visschers 2016). In the appendix, I show that the qualitative results are similar

workers retire with a fixed probability, exiting the market. At the beginning of the next period, a proportional mass of workers is born.

A novel technical contribution of the paper is showing how workers evaluate the set of firms as a multi-armed bandit problem. I show that firms act as *endogenous* bandits. In the standard multi-armed bandit problem, a single decision-maker explores a fixed set of bandits. An important difference in this setting is that multiple workers simultaneously compete over a limited number of positions at firms. The reward from a given firm depends on the probability of hire at that firm. However, the probability of hire is endogenously determined by workers' strategies. *Hiring thresholds* suffice as a manner of describing competitive forces and the ability to learn in equilibrium. Despite the fact that workers simultaneously learn and optimize, thresholds fill a role similar to that of prices in competitive equilibrium. Allowing thresholds to adjust is enough to guarantee an equilibrium exists. While prices and thresholds fill a similar role, the two are not interchangeable. In particular, a worker's choice to apply to a firm does not correspond to demand for that firm in the event the worker is not qualified enough.

Firms act as endogenous bandits whose rewards are determined in equilibrium. Utilizing the multi-armed bandit characterization, I show that aggregate demand over firms satisfies the gross substitutes condition of Kelso and Crawford (1982). This condition enables the development of a threshold adjustment process, wherein thresholds converge to a fixed point equilibrium in which firms' hiring thresholds are consistent with workers' behavior. To the best of my knowledge, my model is the first to endogenously determine the rewards of experimentation through competition in a market setting.[3] Additionally, the techniques easily extend to other markets. I show in the appendix that gradual learning, heterogeneous discounting, and flexible firm capacities can all be incorporated into the model, without affecting qualitative results, as none cause the gross substitutes condition to be violated.

The model can be fruitfully applied to data on workers' tenure that has been presented in labor markets. Two facts have emerged from studies of employment. Workers' transition rates between jobs decrease with age (Menzio, Telyukova, and Visschers 2016), and higher quality workers are more likely to be satisfied with any given match (Network 2017) compared to lower quality workers. I show that both facts are a natural consequence of transient matching with incomplete preference information. Older workers have had more chances to investigate firms, and so are likely to have found a satisfactory match. Then, an older worker's incentive to test out other firms is lower than that of a young worker with far more unexplored options to choose from. As for the second fact, I show that if two workers share a prior regarding their fits at firms, the worker with higher realized fit will search less. Her matches provide more surplus than the

if workers learn upon applying.

[3]For an example with experimentation in a setting with publicly shared information, see Bergemann and Välimäki 1996.

other worker's matches do. If she is not perfectly informed, she may believe her outside options are equivalent to those of the other worker. Then, she wishes to continue searching only if the other worker also wishes to continue searching. As such, she spends weakly less time searching relative to the other worker, and in turn has higher average tenure.

I investigate the impact of commonly suggested policy interventions targeting markets with learning. One such policy is unemployment benefits, as in principle increasing these benefits decreases the cost of search, which allows workers to be more selective. However, when workers have the option of returning to a firm that previously hired them, unemployment benefits also reduce the cost of taking a risk by applying to a competitive firm. When a firm is highly ranked by all workers, increasing unemployment benefits tempers workers' incentives to investigate other firms. In equilibrium, this can lead to underutilization of these firms and decreased surplus.

Next, I characterize the impact of revealing information about a firm's match values. For example, suppose Google held an open house or hired headhunters, informing all workers of their match values at Google. The revelation has two direct effects. Workers who realize that they are poor fits for Google avoid it, while those who are well-suited for Google target it. The workers with lower Google-specific match values strictly benefit from the revelation, while the effect is ambiguous for workers with higher Google-specific match values. As those who are poor fits avoid Google, they intensify the level of competition at other firms. When the market is sufficiently congested, this channel harms workers who would have been good fits at Google.

Last, my model provides insight regarding the influential empirical literature on search (Chade, Eeckhout, and Smith 2017). Models in this literature typically assume firms can add or subtract positions at no cost: if a worker could productively match with a firm, that firm can always add an extra job.[4] In the long run, markets can be expected to adjust, so this is a reasonable assumption. However, on the shorter timescale at which workers make strategic decisions, firms may be unable to freely add and remove positions. In many scenarios, such as when jobs are unionized or during a recession, firms may not be able to respond to an increase in labor supply. In markets with transient matchings, like the described examples, my model shows that calibrated search models which omit the effect of competition can yield upwards biased estimates of worker match quality. To see why, note that the incentive to search is decreasing in the quality of the current match and increasing in the expected value of the outside option. When outside options become less competitive, workers must value their current positions more greatly in order to not switch. Thus, when firms can freely add positions, workers must value their current positions more highly in order to not transfer. Understanding the quality of matches present in equilibrium is critical when evaluating the efficacy of potential

[4]For examples of search models where firms have flexible positions, see Christensen et al. (2005), Menzio and Shi (2011), and Postel-Vinay and Turon (2010).

counterfactual policy changes. As such, systematically biased estimates of match quality can generate incorrect conclusions about the impact of new policies.

The model focuses on markets with non-transferable utility. Despite this, the model applies more broadly to markets where wages are determined independently of match quality. When wages are fixed across realized match values, my qualitative results hold. Many labor markets feature fixed wages. Hall and Krueger (2012) find that less than 35% of job-seekers bargain over wage; most job-seekers accept posted wages. As a result of laws or bargaining, government and union jobs have fixed wages. In France, public school teachers are allowed to apply yearly for placement in any region (Combe, Tercieux, and Terrier 2018). Wages are fixed across regions, conditional on total experience, and so the value of a match depends primarily on the teacher's fit.

I relax the assumption of non-transferable utility in Section 6. To do so, I extend the model to a competitive equilibrium setting, where equilibrium wages are strategically chosen by firms. My qualitative results extend to the transferable utility setting. I also show that resumes—the ability of a worker to prove she has been previously hired elsewhere—play an important role in information transmission. Without them, competitive equilibria may fail to exist.

My results have several implications for designing centralized mechanisms. In many markets, while the original matching is centralized through an algorithm, agents may be free to rematch after their initial assignment. Agents that are aware of this may alter their initial applications accordingly, skewing the initial outcome of the market. In particular, when agents have incomplete information about their match values, and that incomplete information is correlated with their potential for success on the aftermarket, standard algorithms such as deferred acceptance may no longer be strategy-proof. Through understanding how agents with heterogeneous incomplete information match in decentralized markets, we can better understand the impact aftermarkets have on centralized settings.

1.2 Related Literature

This paper relates to several distinct literatures: matching with incomplete information, dynamic matching, directed search, and bandits with collisions.

There is a burgeoning literature on matching with incomplete information. Previous work has focused on settings with centralized clearing houses, whereas in this paper I consider decentralized markets, to better match settings such as labor markets or dating markets. Immorlica et al. (2020) consider school choice where students have incomplete preference information and dynamically learn through costly inspection. They solve the mechanism design problem of generating "regret-free stable" outcomes, wherein agents never regret their search decisions. There are two key differences between their setup and mine. First, they study a centralized one-shot school choice market, with fixed participants. Second, the cost of inspecting a school in their model is fixed and exogenous. Doval (2022), Liu et al. (2014), and Liu (2020) study stable outcomes in markets with incomplete information. Chen and Hu 2020 provide a dynamic justification for stability with incomplete information, in which firms evaluate potential employees according to their "worst" possible match values. These works focus on one-shot matching markets, in which no new participants enter the market after the game begins. In many dynamic environments, including the examples from the previous section, this fails to be the case.[5]

Similarly, recent work in the matching literature has begun to incorporate dynamics.[6] Akbarpour, Li, and Gharan (2020) consider dynamic markets with networked agents, and solve the designer's problem of choosing which agents to match. Anderson and Smith (2010) examine matchings where agents form reputations regarding their quality over time and show that positive assortative matching emerges over time. Ferdowsian, Niederle, and Yariv (2022) are at the intersection of decentralized dynamic matching and matching with incomplete information, and study the hurdles to stability that arise, even in one-shot markets. The paper shows that stringent assumptions are required to ensure stable matchings are equilibrium outcomes in markets with incomplete information. Kadam and Kotowski (2018) also consider markets with transience, and treat the problem from a more classical view of stability. They find conditions under which dynamic stability can be generated in a setting where a centralized authority may be necessary for finding the stable matching. I place more structure on agent's preferences, which enables me to study the related problem in a decentralized environment.

The directed search literature has studied labor markets where workers intentionally target firms.[7] Within the search literature, this paper connects two strains, search with marriage matching and search with frictions. Dagsvik, Jovanovic, and Shepard (1985), Jovanovic (1979), and Miller (1984) consider variations of a single-agent directed-search problem.

Technically, this setting is reminiscent of the multi-armed bandit setting. The solution to the standard multi-armed bandit problem was found by Whittle (1980). Weitzman (1979) studies the mathematical problem where a decision-maker chooses when to stop testing alternatives. Several papers on multi-armed bandits with collisions have recently emerged in the computer science literature, (see Liu, Mania, and Jordan 2020 and Liu et al. 2021). These papers assume that agents know how they are ranked by the other side of the market. In many practical situations, agents face uncertainty regarding their acceptance prospects not only because they are unaware of their competition, but also because they do not know how they will be ranked.

[5] There are also several examples of centralized markets with incomplete information. For instance, Fernandez, Rudov, and Yariv (2022) show that standard predictions of centralized markets are not robust to perturbations of information, while Li, Wang, and Zhong (2016) experimentally tests predictions of truth telling.

[6] For a brief survey of the literature on dynamic matching, see Baccara and Yariv (2021).

[7] Chade, Eeckhout, and Smith (2017) provide a useful survey of the extensive literature studying directed search.

Another key difference between this paper and the literature on bandits with collisions, is that extant algorithms fail to satisfy incentive compatibility when firms disagree on the rankings of workers.

2 The Model

I begin by describing the agents involved. The market matches a continuum of workers with a finite set of firms, $|\mathcal{F}| = F$. Workers are categorized based on observable information, their *class*, and unobservable information, their *type*. A worker knows her class, $c \in \mathcal{C}$, where $\mathcal{C}$ is finite. For instance, a worker knows the school she graduated from, or the grades she received. If a worker had complete information, she would also know her type, $\theta \in c$, describing her match value with each firm. Specifically, θ_w^j is the match value she receives from matching with firm j, while θ_f^j is firm j's value from the match. $\theta = \{(\theta_w^j, \theta_f^j)\}_{j \in \mathcal{F}} \in [0, K]^{2F}$, is a type-$\theta$ worker's vector of match values across firms. Instead of knowing their types, workers know each class's distribution over types. When there are a finite number of types, $m_c(\theta)$ denotes the *total* mass of class-c type-θ workers; otherwise, $m_c(\theta)$ denotes class c's density of θ. The total mass of class c, $m_c = \int_{\theta \in c} m_c(\theta) d\theta$, is the summation of mass across all types of class c. I normalize the total mass of workers to 1: $\int_c m_c dc = 1$.

Each firm j has a hiring capacity of $m(j) > 0$, limiting the mass of workers he can hire each period. A *market* is a triplet $\mathcal{M} = (\mathcal{F}, \mathcal{C}, m)$. I assume that a worker's class already encapsulates any correlation in match values across firms. That is, worker i learning her match value at firm j, is not informed regarding her match value at firm j', conditional on worker i's class. Workers learn about their fit at individual firms, as opposed to learning about their preferences for certain fields or industries. Let $m_c^j(x, y) = \int_{\{\theta | (\theta_w^j, \theta_f^j) = (x, y)\}} m_c(\theta) d\theta$ be the mass of class-c workers with match value (x, y) at firm j. Then, the absence of cross-firm learning can be formally stated as:

$$m_c^j(\theta) = \prod_{j \in \mathcal{F}} m_c(\theta_w^j, \theta_f^j).$$

The matching process is straightforward. Each period, every worker chooses a single firm to apply to. The choice of worker i to apply to firm j in period t is denoted by $a_i(t) = j$. Then, each firm observes his list of applicants; j observes $a_j(t) = \{i | a_i(t) = j\}$, and learns his match value with each applicant, θ_f^j.[8] Firms cannot distinguish between workers of the same class with

[8] The application process can be thought of as containing an interview stage that informs firms. Because firms have previous experience with hiring workers, they are more informed about the quality of the match. Interview frictions have been previously discussed in the matching literature (Lee and Schwarz 2017). To refine the focus on transient matchings in this paper, I abstract from these frictions in the interview process.

equal firm match values. For instance, if two workers of types θ and θ' share a class, apply to firm j in a given period, and $\theta_f^j = \theta_f'^j$; then j must hire workers of the two types with equal probability.

$A_i(t) \in \mathcal{F} \cup \varnothing$ denotes worker i's match in period t, if $A_i(t) = \varnothing$, i is *rejected*. Should a firm j hire a worker, that worker is *accepted*. A type-θ worker accepted by firm j receives θ_w^j, while the firm receives θ_f^j.[9] A rejected worker receives $\theta_w^{\varnothing} = 0$.[10] The market structure makes all matches transient, a worker that wishes to stay at a firm must apply for it every period. The set of workers hired by firm j in period t, is $A_j(t)$. Let $m(A_j(t), \theta)$ denote the mass of type-θ workers in $A_j(t)$. Then, firm j's period-t profit is:

$$\pi_j(t) = \int_{\theta \in A_j(t)} \theta_f^j dm(A_j(t), \theta)$$

Profits are discounted at a rate of δ. Each firm aims to maximize the sum of discounted profits, π:

$$\pi = \sum_{t=0}^{\infty} \delta^t \pi(t).$$

Workers retire with probability $1 - \delta$ at the end of each period, exiting the game. When a type-θ worker retires, a new type-θ worker enters the market. Critically, the new worker will no longer have the information that her predecessor acquired through play. In total, a mass of $(1 - \delta)m_c(\theta)$ class-c type-θ workers are born each period. The fact that workers stochastically exit has two implications. First, a worker's expected utility remains equivalent to the classical discounting interpretation. Second, the distribution of worker-type masses remains unchanged over time, generating a stationary environment.

When a worker enters the market, she is unaware of her exact type, but knows her class. Upon acceptance by firm j, a type-θ worker learns θ_w^j.[11] Rejected workers only learn of their rejection. There is no public history, rather workers can only learn through applying to firms. Workers aim to maximize their lifetime expected utility. That is, worker i of class c, chooses

[9] The main body of the paper focuses on non-transferable utility. As discussed in the introduction, several recent empirical results support this assumption. For instance, Becker (2011) shows that as much as 40% of benefits from employment are non-wage based, implying that proper matching is critical. In Section 6, I extend the model to a transferable utility setting where firms choose match-value dependent wages. I show that the key results carry over to the transferable utility case.

[10] In Section 5.2, I consider the impact of adding unemployment benefits to the market, modelled by $\theta_w^{\varnothing} > 0$.

[11] Other models of learning on the job, such as Jovanovic (1979), use more gradual learning processes, where a worker receives a noisy signal of the true match value. Since the implications generated by gradual learning have already been discussed in the search literature, I instead assume learning is immediate, to isolate the effect of competition. In Appendix section B.1, I show that the core results of this paper still hold when learning is gradual.

how to apply to firms, to maximize:

$$U_i^c = \mathbb{E}_c \left[\sum_{t=0}^{\infty} \delta^t \theta_w^{A_i(t)} \right]$$

To eliminate equilibria that depend upon coordination, or time-specific applications, I refine the set of equilibria to those supported by Markovian strategies. The state variable for a worker is the tuple of her payoff-relevant variables: her class, and her posterior over her type. For a given strategy profile, firms face no payoff-relevant dynamic uncertainty, so their payoff-relevant variable in a given period is the types of their applicants. Let h_t^i denote the private history for a worker i of age t, consisting of her application and realized match values in each period, $h_t^i = ((a_i(1), \theta_w^{A_i(1)}), (a_i(2), \theta_w^{A_i(2)}), \ldots, (a_i(t-1), \theta_w^{A_i(t-1)}))$. For a given history, h_t^i, and strategy profile σ, a class-c worker's posterior regarding her type can be computed using Bayes rule, and is an element of $\Delta\Theta$.

Definition (Markovian Strategies)**.** *A worker i's strategy is Markovian, if in each period, her application is only a function of her class and the posterior over her type. A firm's strategy is Markovian, if in each period, the set of workers hired is a function of the firm's distribution over applicant match values.*

The focus on equilibria in Markovian strategies rules out equilibria that depend on the calendar period. A strategy profile is Markovian if in that profile all agents use Markovian strategies. Similarly, an equilibrium is Markovian if the associated strategy profile is Markovian.

The *state* of the market, is the proportion of workers with each possible posterior over their type. A state is denoted by $\mu = \{\mu_c\}_{c \in \mathcal{C}}$, where $\mu_c \in \Delta\Theta$ denotes the posteriors of class-c workers. Throughout the paper, I focus on the *steady state* generated by a strategy profile, σ—a state that is self-perpetuating when agents follow strategy profile σ. Namely, let the transition map, $u_\sigma : \Delta\Theta \to \Delta\Theta$, denote the mapping from a current state to next period's state when the strategy profile is σ. If for all c, $u_\sigma(\mu) = \mu$, then μ is a steady state.

A Markovian strategy profile is an equilibrium only if no agent has a profitable deviation in the steady state. This amounts to requiring firms to be unable to commit to a hiring policy. A firm's strategy is part of an equilibrium strategy profile, only if in the steady state of that equilibrium, the firm does not wish to deviate in any period *after* observing his applicants. Where appropriate, I drop time indices. Last, to deal with a trivial source of non-uniqueness, this paper considers the matching outcomes that result in the "long-run" for a Markovian strategy profile that constitutes a Perfect Bayesian Equilibrium.

Definition (Outcomes)**.** *An outcome for market $\mathcal{M}$, of a Markovian strategy profile σ, is the distribution over each types' applications to firms in the steady state of σ.*[12]

An outcome is the payoff-relevant information for a snapshot of the market. Namely, outcomes include the proportion of information each worker has learned, as well as their choice of firms to apply to. Restricting attention to the outcomes of Markovian strategy profiles removes another trivial source of non-uniqueness. Should a worker have two strategies which differ conditional on information regarding a firm j that she never applies to under either strategy, the outcomes of the two strategies are equivalent.

3 Incomplete Information Generates Transience

I begin by analyzing the benchmark case where workers are fully informed about their type. I find that market outcomes do not exhibit transience, instead workers follow static application strategies. In this section, I detail how the complete information setting exemplifies the interplay between transience and information. Should an outside observer compare outcomes in a complete information market with transient matchings to outcomes in a market with permanent matchings, she would be unable to distinguish them. In the following section, I show that the complete information case is similar to the case where long-lived workers have incomplete preference information. However, several key differences emerge between the two. With incomplete information, worker outcomes exhibit path dependence—the results of a finite number of initial applications determine their long-term earnings. Furthermore, when markets are congested, workers may be persistently matched to firms different from those they would be matched to under complete information.

To begin, I develop a benchmark for the case where workers are fully informed, and workers and firms agree on the value of each match. That is, match values are *aligned*.[13] Match values are aligned if they can be represented by a joint "ordinal potential." This is similar to the notion of potentials in normal-form games (Monderer and Shapley 1996). Formally, alignment is satisfied if there exists a matrix $\Phi = (\Phi_{\theta j})_{j\in\mathcal{F}} \in \mathbb{R}$, such that for any types $\theta, \theta' \in \mathcal{C}$ and firms j, j':

$$\text{If } \theta_w^j > \theta_w^{j'} \text{ then } \Phi_{\theta j} > \Phi_{\theta j'} \text{ and if } \theta_f^j > \theta_f'^j \text{ then } \Phi_{\theta j} > \Phi_{\theta j'}$$

For instance, if workers and firms split the surplus from any match according to a fixed

[12] I show in Section 4.1 that every Markovian strategy profile generates a unique steady state, and therefore this notation is well-defined.

[13] The term aligned comes from Ferdowsian, Niederle, and Yariv (2022), in which alignment is shown to be critical for the emergence of stability in decentralized markets.

proportion, match values are aligned: $\exists \alpha$ s.t. $\forall j, \theta : \theta_w^j = \alpha \theta_f^j$.[14] Alignment will imply that a firm cannot reject a worker and trigger a cycle of other firms' rejections, which culminates in a preferred worker's application.[15] When match values are aligned, I will simplify notation by letting, $\forall \theta, j\ \theta_w^j = \theta_f^j = \theta^j$, but the results hold more generally for any aligned match values.

Definition (Complete Information Market)**.** *A market $\mathcal{M}_I$ is a complete information market, if every worker class c contains a single type.*

In a complete information market, all workers are perfectly informed about their match values at each firm. Suppose all match values are strictly ordered. Then, there exists some class-c^* and firm j^*, such that class-c^* workers and j^* generate the maximal match value: $(c^*, j^*) = \arg\max_{c,j \in \mathcal{E}} \theta_c^j$. Class-$c^*$ workers must apply to j^* with positive probability. Otherwise, some class-c^* worker, i, would benefit by deviating and applying to j^*. Because no other class-c^* workers are applying to j^*, i is guaranteed to be hired since j^* prefers i to any other applicant. Furthermore, by switching her application, i will receive a higher match value than she would have previously. Indeed, either all class-c^* workers must apply to j^*, or at least enough must apply to force j^* to begin rejecting class-c^* applicants. If too many class-c^* workers apply to j^*, the resulting competition may leave class-c^* workers preferring their second choice, $j_2 = \arg\max_{j \neq j^*} \theta_{c^*}^j$. Firm j^* is *oversubscripted* if class-c^* workers would prefer guaranteed hire by j_2 to applying to j^* when all other class-c^* workers apply to j^* as well: $\frac{m(j^*)}{m(c^*)} \theta_{c^*}^{j^*} < \theta_{c^*}^{j_2}$. In the event of oversubscription, at least $m(j^*) \frac{\theta_{i^*}^{j^*}}{\theta_{c^*}^{j_2}}$ class-c^* workers must apply to j^*.

If oversubscription occurs, j^* is filled to capacity, otherwise all class-c^* workers apply to j^*. In either event, a submarket can be considered with strictly less agents than $\mathcal{M}_I$. The previous argument can be repeated in the submarket, removing another agent from the market. Extra care must be taken if a class or firm that was previous part of an oversubscription is being considered. I go into detail about how to ensure indifference in such a step, when I formalize the procedure as the "Top-Down" algorithm in the appendix. Because there are a finite number of classes and firms, the process must eventually terminate, in at most $|\mathcal{C}| + |\mathcal{F}|$ steps.

Lemma 1 (Complete Information Equilibrium)**.** *The Top-Down algorithm characterizes the unique equilibrium outcome of $\mathcal{M}_I$ when match values are aligned.*

Matchings are permanent in complete information environments—workers apply to the same firms every period. However, despite information being complete, decentralization inherently

[14] In this case, $\Phi = (\Phi_{\theta j})_{\theta \in \mathcal{C}, j \in \mathcal{F}}$ where $\Phi_{\theta j} = \theta_f^j$ for all θ, j serves as an ordinal potential.

[15] In Voorneveld and Norde (1997), it is shown that potential games cannot have cycles in the payoff matrix. In this setting, their result will imply that markets with aligned match values cannot have rejection cycles. That is, in equilibrium, no firm can reject a worker, and thereby trigger a change in applications such that the firm receives an application from a preferred worker.

generates two types of inefficiencies. First, workers that are qualified for desirable positions overcompete for those positions. This feature of oversubscription generates significant congestion, a negative externality on other workers of their class. Second, workers do not consider their impact on other classes of workers. For instance, suppose there are two firms, $\mathcal{F} = \{j_1, j_2\}$, each with a capacity of 1/2. There are two classes of workers, $\mathcal{C} = \{c_1, c_2\}$, with match values $\theta_1 = (5, 4)$ and $\theta_2 = (4, 0)$, and masses $m_{c_1} = 1/2, m_{c_2} = 1/2$. Then, in equilibrium, class-c_1 workers will always apply to j_1, blocking class-c_2 workers from being hired there. From a utilitarian standard point, a more efficient outcome would involve class-c_1 workers applying to j_2, while class-c_2 workers apply to j_1. Inherently, workers do not consider the externality they impose on workers of other classes. As will be seen, introducing incomplete information adds a third externality. Workers do not consider how their applications block other workers from searching for good fits. In Section 4.5, I show that when workers have incomplete information and become more long-lived, outcomes converge to the complete information outcomes, though marked differences appear for individual workers.

4 Equilibrium Characterization

When workers have incomplete information and are not long-lived, they face a non-trivial trade-off. Workers must decide between exploiting their information—applying to firms with high expected match values; and learning—applying to new firms. When determining whether to explore, workers must take into account other workers' application decisions, which generate competition over firms.

I proceed by characterizing best responses for both firms and workers. First, the following section shows that in any strategy profile where a firm does not hire his most qualified applicants, he has a profitable deviation. Once firms' strategies are thresholds, workers can anticipate those thresholds in equilibrium, and evaluate firms accordingly. Using techniques from the literature on multi-armed bandits, worker behavior can then be determined. Then, because workers evaluate firms as if they were bandits, demand for a firm is increasing in other firms' thresholds. Using this, I develop an algorithm to find a fixed point where the demand for each firm coincides with the firm's capacity and threshold, and show that the algorithm captures the unique equilibrium outcome. Last, I show that worker's search patterns aligns with empirical results.

Before proceeding, Lemma 2 shows that any Markovian strategy profile generates a unique outcome, implying that the focus on steady state outcomes is well-defined.

Lemma 2 (Unique Steady State). *Any Markovian strategy profile has a unique steady state.*

4.1 Hiring Thresholds

In Example 3, it was assumed that a firm would always hire his most qualified applicants in every period, without regard for how doing so would impact future applications. I show that firms' equilibrium strategies can be described as threshold strategies.

When a firm's strategy profile dictates he rejects workers more qualified than those he accepts, he has a profitable deviation. Through accepting workers with higher match values, the firm is able to reap the benefit from hiring the qualified worker immediately, instead of delaying the match value from acceptance to a future period. The key insight is that when match values are aligned, a firm which rejects a worker with a high match value can only do so to incentivize that worker to apply again in a future period. However, the rejection simply shifts back the expected gains from matching with that worker, and so the firm would prefer to accept her in the current period.

Proposition 1 (Firms Use Hiring Thresholds)**.** *Suppose match values are aligned. In the steady state of any equilibrium, if firm j hires a worker with a match value of θ_f^j, then firm j also hires all applicants with match values above θ_f^j. Additionally, if firm j hires below its capacity, $m(A_j) < m(j)$, then j hires all applicants.*

A worker's expected utility from a firm inherently depends upon that firm's hiring decision, which in turn is a function of the application strategies of other workers. There is a natural ordering on worker types at firm j, $\theta \succsim \theta'$ if $\theta_f^j > \theta_f'^j$. Proposition 1 implies that the firms' hiring decision can be summarized by the minimal hired θ_f^j, and the probability with which type-θ workers are hired. For a worker, a firm's strategy is payoff-relevant only in how it affects the probability of hire at that firm. This motivates a natural method of summarizing a firm's strategy, the threshold in match values below which it rejects applicants.

Definition (Hiring Threshold)**.** *A hiring threshold for some firm, j, is a tuple (v_j, p_j), which consists of a match value and a hiring probability.*

Crucially, in the steady state of an equilibrium, thresholds are time-invariant. To determine firm j's threshold, begin by finding the set of workers that apply to j within a single period. Because firm j uses a threshold strategy, j hires a worker with the minimal match value. Let θ be that worker's type. Set v_j to θ's firm j match value, θ_f^j, and p_j to type-θ's probability of hire at j. If the mass of applicants received by j is below its capacity, $m(a_j) \leq m(j)$, define j's threshold as $(0, 1)$—all applicants to j are hired in equilibrium.

A threshold suffices to describe a firm's behavior in equilibrium. Workers can use v_j and p_j to compute their expected utility from applying to firm j. If, for a worker type θ, θ_f^j is above

v_j, then type-θ workers are always hired by j. If θ_f^j is below v_j, type-θ workers are never hired by j. Last, if $\theta_f^j = v_j$, then type-θ workers are hired by j with probability p_j.

It will be convenient to rank thresholds in the lexicographic order. A threshold, (v_j, p_j), is greater than another threshold, (v'_j, p'_j), $(v_j, p_j) \succ (v'_j, p'_j)$ if $v_j > v'_j$, or if $v_j = v'_j$ and $p_j \geq p'_j$. $(v, p) = \{(v_j, p_j)\}_{j \in \mathcal{F}}$ will refer to the vector of thresholds.

The optimality of threshold strategies is a direct consequence of firms' inability to commit to hiring policies. If a firm could commit to hiring workers of a given class, then that firm could encourage certain classes of workers to apply through affirmative action policies. This would enable the firm to attract highly qualified workers of types within those classes through the promise of guaranteed jobs. I expand on this motive in Section 5.1.

4.2 Equilibrium Worker Strategies

Next, I discuss the characterization of a worker's equilibrium strategy. As previously mentioned, workers trade-off exploring firms where their match value is still unknown, and exploiting firms that are known to be profitable matches. I discuss a technique from the multi-armed bandit literature, which provides an index-based solution method that holds for any market where all firms hire all applicants. I extend the technique to settings where the hiring probabilities are endogenously determined, and show that the index solution carries over.

Definition (Gittins Index)**.** *(Gittins 1979) The Gittins index for worker i at firm j given history h_t^i, $GI_i(j, h_t)$, is the solution to the following optimal stopping problem, where τ can depend upon the realization of match values:*

$$GI_i(j, h_t^i) = \sup_{\tau} \frac{\mathbb{E}_{i,h_t^i}\left[\sum_{t=1}^{\tau} \delta^t \theta_w^j\right]}{\mathbb{E}_{i,h_t^i}\left[\sum_{t=1}^{\tau} \delta^t\right]}$$

Gittins indices provide a simple characterization of the benefits from learning. Intuitively, the Gittins index of firm j for a class-c worker is her expected match value at j, weighted by her ability to learn. Namely, the worker can strategically reapply to j conditional on the realized match value. If the worker discovers a high match value at j she can take advantage of the positive realization and reapply to j, instead of leaving immediately as she would after discovering a low match value at j. The ability to strategically reapply biases the Gittins index upward from the standard expected value, and incorporates a benefit from learning. When describing the Gittins index of a newly arrived worker, I omit the trivial match history to conserve space.

Determining the optimal worker policy can be done tractably, thanks to a result from the operations literature. Rather than a worker worrying about the timing of learning and equilibrium effects, the worker simply compares the equilibrium Gittins indices associated with each firm. Naturally, these Gittins indices are endogenous objects, which depend on the proportion of workers applying to each firm.

The above formulation is insufficient when firms do not hire all applicants. However, declaring firm j's threshold allows for the computation of each worker's initial Gittins index at j. To do so, simply replace each match value with the imputed match value condition on (v_j, p_j). Suppose for an equilibrium strategy profile σ, the hiring thresholds are (v, p). Let $\psi_\theta(v, p)$ be a type-θ worker's realized match values given thresholds (v, p). That is:

$$\psi_\theta^j(v,p) \equiv \begin{cases} \theta_w^j & \theta_f^j > v_j \\ p_j \theta_w^j & \theta_f^j = v_j \\ 0 & \theta_f^j < v_j \end{cases}$$

What stopping strategy τ determines j's Gittins index? It can be shown that the following heuristic determines the value of $GI_i(j, h_t^i)$. Choose an arbitrary value x and apply to j. If the resulting match value is above x, stay with j forever, $\tau = \infty$. If the match value is below x, leave j immediately, $\tau = 1$. This procedure determines a value for the stopping problem if x is equal to the value of the stopping problem. The maximal value of x determines the Gittins index. Lemma 3 formalizes this heuristic computation.

Lemma 3 (Firms' Equilibrium Indices)**.** *$GI_i^\sigma(j, h_t^i)$ for a class-c worker i is characterized by the fixed point solution to the following functional equation:*

$$GI_i^\sigma(j,h_t^i) = \frac{\mathbb{P}\left[\psi_w^j < GI_i^\sigma(j,h_t^i)\right]\mathbb{E}\left[\psi_w^j|\psi_w^j < GI_i^\sigma(j,h_t^i)\right] + \frac{1}{1-\delta}\mathbb{P}\left[\psi_w^j \geq GI_i^\sigma(j,h_t^i)\right]\mathbb{E}\left[\psi_w^j|\psi_w^j > GI_i^\sigma(j,h_t^i)\right]}{\mathbb{P}\left[\psi_w^j < GI_i^\sigma(j,h_t^i)\right] + \frac{1}{1-\delta}\mathbb{P}\left[\psi_w^j > GI_i^\sigma(j,h_t^i)\right]}$$

In Example 3, determining the value of $GI_i(f_s)$ was trivial.[16] Since each worker knows her match value at f_s, for any choice of τ, $GI_i(f_s) = 2$. Computing $GI_i(f_r)$ requires slightly more work. First, consider the case where $m(f_r) = 1$, that is, f_r is not capacity constrained. Then, f_r must hire all applicants in equilibrium. If a worker applies to f_r and receives a match value of $1/2$, she immediately knows her type is θ_l. If she applies to f_r a second time, the numerator of $GI_i(f_r)$ is increased by $1/2\delta$, and the denominator is increased by δ. If $GI_i(f_r) > 1/2$, then it can be shown that the solution of the stopping time problem induced by $GI_i(f_r)$ never involves applying a second time to f_r after observing $\theta_w^{f_r} = 1/2$. If the worker had applied to f_r and received a match value of 3, she would know that her type was θ_h. An extension of the previous

[16] As a reminder, both types had a match value of 2 at f_s, while type-θ_h had a match value of 3 at f_r and type-θ_l had a match value of $1/2$ at f_r.

logic shows that if $GI_i(f_r) < 3$, the solution to the stopping time problem involves applying to f_r a second time upon observing $\theta_w^{f_r} = 3$. Putting everything together, the value of $GI_i(f_r)$ is found by infinitely applying to f_r, if type θ_h, and applying only once if type θ_l, yielding:

$$GI_i(f_r) = \frac{1/2\left(\frac{3}{1-\delta} + 1/2\right)}{1/2\left(\frac{1}{1-\delta} + 1\right)}$$

Notably, when computing $GI_i(f_r)$, the distribution over possible match values at f_s was not utilized. Even when the number of firms, F, increases, the number of computations to determine all Gittins indices also increases linearly in F, one additional step for each additional firm.

Standard results in the operations literature show that, determining the optimal policy takes two steps: in each period compute the Gittins index for each firm, then apply to the firm with the highest Gittins index (Whittle 1980). Previous results involving directed search settings or Gittins indices have required that only one decision-maker is present or that the decisions made by each decision-maker are independent of one another.[17] Through classifying firms' strategies as threshold strategies, the results are extended to settings where multiple workers compete over matches.

4.3 Alignment Implies Uniqueness

In this section, I show that when firms and workers agree on the quality of each match, i.e., match values are *aligned*, there exists a unique equilibrium outcome under a mild condition on payoffs. To do so, I first develop the following example, which illustrates the specificity in payoffs required to generate non-unique equilibrium outcomes.

Example 1 (Non-Unique Outcomes)**.** *Rows represent worker types, while columns represent firms. The mass of each agent is given in the first element of the corresponding row or column. The match values for a worker and firm are given at the intersection of their row and column.*

Firms / *Class-c_1 Workers*	$m(f_r) = 1/2$	$m(f_s) = 1/2$
$m_1(\theta_h) = 1/4$	$(4, 4)$	$(3, 3)$
$m_1(\theta_l) = 1/4$	$(2, 2)$	$(3, 3)$

[17] For examples in economics, see Papageorgiou 2018, Rothschild 1974, Urgun 2021, and Weitzman 1979.

Firms / *Class-c_2 Workers*	$m(f_r) = 1/2$	$m(f_s) = 1/2$
$m_2(\theta) = 1/2$	$(1, 1)$	$(1, 1)$

There are two worker classes, $\mathcal{C} = \{1, 2\}$, two firms, $F = 2$, and $\delta = 0$. When class-c_1 workers are guaranteed to be hired, and indifferent over firms, there exists a multiplicity of equilibrium outcomes.

Class-1 workers are indifferent between the two firms, and furthermore are never rejected in equilibrium. Therefore, incoming class-1 workers are indifferent between the two firms in any equilibrium. There exists a continuum of equilibrium outcomes characterized by the proportion of class-1 workers that applies to f_r.

The non-uniqueness of equilibrium outcomes cannot be trivially fixed by requiring each firm's Gittins index to be unique in the absence of firm hiring capacities. To see why, return to Example 1 and suppose class 1 contained several additional types, such that match values below 1 were possible at both firms. In the absence of firm hiring constraints, class 1's Gittins index at f_r, $GI_1(f_r)$, would be higher than $GI_1(f_s)$. In practice, the difference would not factor into class-1 workers' decision-making, because in any equilibrium, workers with match values below 1 are never hired.

Furthermore, Example 1 did not hinge on the absence of learning. Consider the same market, with some $\delta > 0$. A simple computation can be used to evaluate $GI_1(f_r) = \frac{1/2\frac{4}{1-\delta}+1/2\cdot 2}{1/2\frac{1}{1-\delta}+1/2\cdot 1}$ while $GI_1(f_s) = 3$. For $\delta > 0$, $GI_1(f_r) > GI_1(f_s)$ due to the option value of switching to f_s if a low utility is realized. However, a small adjustment to θ_h^r can set $GI_1(f_r) = GI_2(f_s)$. Replacing $\theta_h^{f_r} = 4$ with $\theta_h^{f_r} = 4 - \delta$ ensures that the two Gittins indices are equal once more. In the resulting market, there again exists a continuum of equilibrium outcomes. As such, a more stringent condition is needed to rule out this trivial form of non-uniqueness.

The *strict dynamic preferences* assumption requires workers to not face indifference between two firms when each firm hires deterministically. This assumption depends on δ, but only rules out a finite number of parameters from an infinite set. The parameter set after strict dynamic preferences are ensured is generic. For instance, the parameter set which generated multiple equilibrium outcomes in Example 1 is eliminated by strict dynamic preferences, preventing class-1 workers from mixing. Intuitively, strict dynamic preferences requires that the equilibrium Gittins indices at any two non-competitive firms cannot be equal. It should be emphasized that this assumption is exogenous, the assumption only depends on utilities and masses.

Assumption 1 (Strict Dynamic Preferences)**.** *Let every type of worker class c have probability of hire 0 or 1 at firm j and j'. Strict dynamic preferences holds if for every class c there does not exist a type $\theta \in i$ such that the following hold:*

- *θ has probability 1 of hire at j*
- *$GI_i(j) = GI_i(j')$, $\theta_w^j = GI_i(j')$, or $\theta_w^j = \theta_w^{j'}$*

First, no strict dynamic preferences implies that two strategy profiles, which induce different equilibrium outcomes, must also generate different hiring thresholds. The claim is formally proved in the appendix. Given this, Proposition 2 shows that equilibrium outcomes must be unique under strict dynamic preferences.

Proposition 2 (Unique Outcome)**.** *When match values are aligned and satisfy strict dynamic preferences, there is a unique equilibrium outcome.*

To provide intuition, suppose σ and σ' are distinct equilibrium outcomes. Then, the thresholds induced by σ and σ' must be distinct. Without loss of generality, let firm j have a higher threshold under σ than σ', $(v_f(\sigma), p_f(\sigma)) \succ (v_f(\sigma'), p_f(\sigma'))$. Then, f must have more high quality applicants under σ relative to σ', otherwise it could not have a higher threshold. Those applicants must come from another firm f' under σ'. For those applicants to switch to f, f' must be less attractive under σ. However, when match values are aligned, thresholds imply that the quality of workers hired is monotonic, $(v_{f'}(\sigma), p_{f'}(\sigma)) \succ (v_{f'}(\sigma'), p_{f'}(\sigma'))$. Iterating this logic implies that there exists a set of firms with higher thresholds. Since all involved firms have binding capacity constraints, weakly more workers must be hired under σ. Furthermore, the fact that all hiring thresholds are higher, implies that total worker utility is also higher. Then, at least one class, c, must be strictly better off under σ. Since hiring thresholds were lower under σ', class-c workers can profitably deviate to their strategy under σ. Then, σ' could not have been an equilibrium outcome.

The uniqueness of equilibrium transient matchings is useful for the prospective econometrician. When estimating structural parameters, the statistician does not need to worry about the problem of equilibrium selection. Instead, the unique equilibrium can be determined and exploited.

4.4 Gross Substitutes in Equilibrium

In this section, I proceed by describing equilibrium strategies, utilizing firm thresholds to characterize equilibria. Importantly, because workers evaluate firms as if they were bandits, their demand for firms is increasing in other firm's thresholds.

First, I show that standard bandit results can be extended from settings where all firms have unbounded capacities to equilibrium settings where firms can only hire a limited subset of workers.

Lemma 4 (Firms are Endogenous Bandits)**.** *For any equilibrium strategy profile σ, for any private history h_t^i, each worker i applies to $j \in \arg\max_j GI_i^\sigma(j, h_t^i)$.*

Lemma 4 shows that the optimal worker strategy for a given set of thresholds can be simply determined. For each worker i, determine each firm's Gittins index, potentially updating the probability of each match value based on h_t^i. Each worker i applies to a firm with maximal Gittins index, $j \in \arg\max_j GI_i^\sigma(j, h_t^i)$. This application strategy prescribes a distribution over learning paths for each worker type. In turn, the mass of informed worker types and therefore the proportion of each type applying to each firm is fixed, as well as their equilibrium hiring rates. Type-θ's demand for firm j given thresholds (v, p) will be denoted $D_j^\theta(v, p)$. *Aggregate demand*, $D(v, p)$ is defined as the integral over all types' realized demands.

Definition (Aggregate Demand)**.**

$$D(v,p) = \left(\int\limits_\theta D_{j_1}^\theta(v,p)d\theta, \int\limits_\theta D_{j_2}^\theta(v,p)d\theta, \ldots, \int\limits_\theta D_{j_F}^\theta(v,p)d\theta \right)$$

The characterization of optimal index-based worker strategies has an additional feature. When a firm increases its threshold, demand at other firms can only increase. To see why, note that workers apply to the firm with the maximal Gittins index. Increasing a firm's threshold decreases that firm's Gittins index for all types. Furthermore, the independence of the Gittins indices implies that all other firms' Gittins indices remain fixed. The increase in the threshold will not change other firms' demands unless it causes the Gittins index of one type at a firm to exceed their Gittins index at another firm. Doing so, will shift that type's demand to the other firm. It can be shown that the effect on that type's demand in future periods does not exceed the immediate effect. More generally, increasing the thresholds of a subset of firm cannot decrease the demand of firms outside that subset. Namely, aggregate demand satisfies the *gross substitutes* condition, used in Kelso and Crawford (1982) and Gul and Stacchetti (2000), to prove existence of equilibria. Let firm j's demand, $D_j(v, p) = \int\limits_\theta D_j^\theta(v, p)d\theta$, be the demand across all types for firm j. Then, gross substitutes is defined as follows.

Definition (Gross Substitutes)**.** *Let $\boldsymbol{F}$ be a set of firms, $\boldsymbol{F} \subset \mathcal{F}$, and $(v, p), (v', p')$ be two thresholds, such that $(v_j, p_j) = (v'_j, p'_j)$ for $j \in \mathbf{F}$ and $(v_j, p_j) \geq (v'_j, p'_j)$ otherwise. Gross substitutes is satisfied if demand for $\boldsymbol{F}$ is greater under (v_j, p_j) than (v'_j, p'_j):*

$$\forall j \in \boldsymbol{F} : D_j(v, p) \geq D_j(v', p')$$

Proposition 3. *Aggregate demand satisfies gross substitutes.*

Given Proposition 3, the equilibrium can be found through the *threshold adjustment process*. The the threshold adjustment process follows five steps:

Step 1 (Remove Capacities): Treat every firm as if it had no capacity constraint: set all thresholds to $(v_j, p_j) = (0, 1)$.

Step 2 (Compute Demand): For every worker class, compute the Gittins index of each firm. Using these Gittins indices, determine individual demand, and in turn, aggregate demand for each firm.

Step 3 (Select Firm): If all firms' have demand below capacity, $\forall j, D_j(v, p) < m(j)$, terminate. Otherwise, select a firm j such that j's aggregate demand exceeds its capacity, $D_j(v, p) > m(j)$.

Step 4 (Reduce Capacity): Decrease the rate of hire for workers with the lowest match value at firm j continuously, $p_j \to 0$. If p_j reaches zero, repeat with the next lowest match value at firm j. Stop when either enough workers are fired, or decide to apply to other firms, $D_j(v, p) = m(j)$. Importantly, Proposition 3 implies that increasing firm j's threshold never causes a decrease in demand for other firms.[18]

Step 5 (Repeat or Terminate): If there exists another firm j' such that $D_{j'} > m(j')$: repeat step 3. While doing so, whenever a type-θ worker with a history realization that occurs with positive probability, would be rendered indifferent between j' and one or more previously treated firms, $\mathbf{F}$, simultaneously decrease the p_j of all $j \in \mathbf{F} \cup j'$, to ensure that the type's Gittin indices are equal across all such firms, $GI_\theta(j) = GI_\theta(j')$, while reallocating type-θ workers across these firms to equalize demand and capacity. This continues until either all $p_j = 0$, or $D_{j'}(v, p) = m(j')$.[19]

[18]This step requires a technical adjustment when worker types have positive mass to avoid cycles, see the appendix for the details.

[19]This characterization of equilibrium behavior is similar in spirit to the characterization of Azevedo and Leshno (2016). There are two key differences. First, aggregate demand is not only a function of workers' initial applications, but also must account for applications from workers who have previously applied and learned. Second, my formulation allows for settings where firms are indifferent over a positive mass of workers—for instance, when worker types are discrete.

By design, the procedure begins with all workers applying as if they anticipated firms would not be congested. In practice, this will cause certain firms to receive more applicants than could be hired, due to the firms' capacity constraints. Through continuously firing the worst applicants, a firm decreases the number of applicants who expect to be hired. The decreased rate of hire may cause workers to apply to other firms, but due to the gross substitutes condition, other firms' demands never decrease. Eventually, the firm reaches its capacity. The process then repeats with another firm that is hiring more workers than it can support. However, any time a worker is rendered indifferent between the current firm and a previously treated firm, that worker then randomizes between the two firms so as to not violate the previously treated firm's capacity. This ensures that once a firm has met its capacity, it never again exceeds its capacity. As such, the threshold adjustment process terminates in a finite number of iterations, one for each firm.

Theorem 1 (Equilibrium Characterization)**.** *The threshold adjustment process culminates in an equilibrium strategy profile in at most F steps.*

Standard bandit techniques such as the Gittins index are normally only valid in decision problems. Despite this, the characterization through hiring thresholds provides the necessary step for the proof of Theorem 1. These thresholds reduce the best response problem into worker-specific decision problems, each of which can then be solved through computation of Gittins indices for a given set of thresholds. Furthermore, this procedure only requires a finite number of iterations, making implementation simple.

The threshold adjustment process shares several features with the worker proposing Deferred Acceptance algorithm. Workers begin by proposing to their ideal positions, then reconsider as they are rejected. In settings where match values are not aligned, there may be multiple equilibria. However, the equilibria can be ranked through a weak order in terms of worker and firm preferences. Consider two equilibrium strategy profiles, σ and σ', where the thresholds induced by σ are greater than those induced by σ', $(v,p)(\sigma) \geq (v,p)(\sigma')$. Workers must prefer the equilibrium under σ' to σ, while firms prefer σ to σ'.

Proposition 4 (Threshold Ranking)**.** *If σ and σ' are equilibrium strategy profiles, and $(v,p)(\sigma) \geq (v,p)(\sigma')$, then for all $c \in \mathcal{C}, U^c(\sigma) \leq U^c(\sigma')$, and for all $j \in \mathcal{F}$, $\pi_j(\sigma) \geq \pi_j(\sigma')$.*

Importantly, if the expected utility under σ was above that under σ', a class c worker could deviate to her strategy under σ, be hired weakly more often, and receive greater utility. Conversely, firms receive higher utility under σ because their thresholds are higher, implying a higher quality for each worker hired.

4.5 Efficiency and Long-Lived Behavior

In this section, I show how congestion can lead to inefficiently low levels of search. I then prove that search converges to the efficient benchmark as δ converges to 1. In particular, outcomes converge to the complete information outcome as δ converges to 1. However, in certain markets, for any $\delta < 1$, congestion always prevents a mass of workers from learning their type—for any $\delta < 1$, these workers would strictly benefit from learning their type at the outset.

In Example 2, two classes of workers decide whether to apply to f_r, or take a guaranteed payoff at f_s. I show that in equilibrium, f_r would prefer a certain class of workers to apply, but those workers do not do so, due to the fear of rejection. Had a social planner determined the acceptances of workers, total equilibrium payoff would be higher, and furthermore, f_r would have been better off. Subsequently, I show that through hiring headhunters, f_r can attract these workers, increasing its equilibrium profit. The safe firm has enough capacity to hire all workers, and so hires all applicants in equilibrium. The following payoff matrix depicts the market.

Example 2 (Congestion Restricts Efficient Search)**.**

Class-c_1 Workers \ *Firms*	$m(f_r) = 1/2$	$m(f_s) = 1$
$m(\theta_h) = 1/4$	$(3, 3)$	$(2, 2)$
$m(\theta_l) = 1/4$	$(1/2, 1/2)$	$(2, 2)$

Class-c_2 Workers \ *Firms*	$m(f_r) = 1/2$	$m(f_s) = 1$
$m(\theta) = 1/2$	$(2, 2)$	$(2 - \epsilon, 2 - \epsilon)$

There are two classes of workers, $\mathcal{C} = \{c_1, c_2\}$. For $\delta \in (1/3, 1/2)$, congestion restricts the willingness of c_1 workers to search, resulting in an inefficient outcome.

In Example 2, class-c_1 workers face uncertainty regarding their type, and would prefer to apply to f_r in the absence of congestion for sufficiently high δ. However, the presence of class-c_2 workers means that a class-c_1 type-θ_l worker is not hired at f_r. For $\delta < 1/2$, the only equilibrium is the strategy profile where all class-c_1 workers apply to f_s, while all class-c_2 workers apply to f_r. To see why, consider the payoff a class-c_1 worker receives from deviating and applying to f_r. With probability 1/2, her type is θ_h. She receives 3 immediately from the match, and can continue applying to f_r indefinitely. Otherwise, her type is θ_l, causing f_r to reject her, after which she applies to f_s indefinitely. In expectation, her utility from deviation is $1/2\frac{3}{1-\delta}+1/2\frac{2\delta}{1-\delta}$, whereas if she had remained at f_s, she would have received $\frac{2}{1-\delta}$. For $\delta < 1/2$, this deviation is not profitable, and so the original strategy profile is an equilibrium.

However, this equilibrium is inefficient for intermediate values of δ, namely when $\delta \in (1/3, 1/2)$. Suppose a centralized authority required f_r to hire all applicants from class-c_1. Then, class-c_1 workers who apply to f_r receive $1/2\frac{3}{1-\delta}+1/2(1/2+\frac{2\delta}{1-\delta})$ and so apply to f_r when $\delta > 1/3$. In the new equilibrium, all class-c_1 workers would initially apply to f_r, then type-θ_h workers would continue applying to f_r while type-θ_l workers would apply to f_s. This implies that f_r receives a mass of $(1-\delta)(\frac{1/4}{1-\delta}+1/4) = 1/4(2-\delta)$ class-c_1 applicants. Importantly, for any $\delta \in (1/3, 1/2)$, $1/4(2-\delta) < 1/2$, so f_r does not exceed its capacity constraint. In equilibrium, the applications of class-c_2 workers to f_r would then adjust accordingly to render class-c_2 workers indifferent between the two firms. From a utilitarian perspective, when $\delta \in (1/3, 1/2)$, this equilibrium is more efficient than the previous one. Indeed, while the new total worker utility is above $1/4\frac{3}{1-\delta} + 1/4(1/2 + \frac{2\delta}{1-\delta}) + 1/2(2-\epsilon)$. Note that class-$c_1$ workers necessarily receive more than 2 in utility, otherwise they would be better off applying to f_s. Therefore, total worker utility has increased from 2, the previous total worker utility, for sufficiently small values of ϵ.[20] Total welfare is increased through preventing congestion from limiting socially optimal search.

Next, I investigate outcomes when workers are long-lived. As δ increases, the inefficiency found in Example 2 is ameliorated. Indeed, for sufficiently high values of δ, workers are motivated to search for their maximal match values. In fact, the equilibrium outcome converges to the complete information equilibrium outcome as workers become more patient, i.e., when δ converges to 1. The inefficiencies present in the complete information environment carry over to the case where workers are long-lived, and have incomplete preference information. When workers do not know their type, they must also invest time to determine their top possible match. However, workers may be unwilling to do so if δ is sufficiently small. In the limit, this source of inefficiency converges to 0, as δ goes to 1. However, for any $\delta < 1$, the inefficiency always remains strictly positive in markets with oversubscription. Even as δ goes to 1, certain worker types always benefit from learning their match values, i.e., there exists path dependence despite vanishing time frictions. Path dependence arises due to the possibility that an unlucky hiring realization can leave a worker uninformed regarding her type. Example 3, with a single class of workers, $\mathcal{C} = 1$, illustrates this form of path dependence.

[20]It is also worth noting that f_r benefits from this change as well. Thus if f_r could commit at the beginning of each period to hiring all class-c_1 workers, he would prefer to do so.

Example 3 (Path Dependence)**.**

Firms / *Workers*	$m(f_r) = 1/4$	$m(f_s) = 1$
$m(\theta_h) = 1/2$	$(3, 3)$	$(2, 2)$
$m(\theta_l) = 1/2$	$(1/2, 1/2)$	$(2, 2)$

Even when workers are long-lived, path dependence emerges in equilibrium.

The set of possible equilibrium strategy profiles can be categorized by the number of failed applications to f_r after which workers cease applying to f_r, and instead apply to f_s forever. Naturally, the equilibrium strategy profile depends upon δ. When δ is low, workers are unwilling to invest into learning, and all apply to f_s. However, even when δ is high, any strategy profile where all type-θ_h workers learn their type with arbitrarily high precision through repeated applications to f_r cannot be an equilibrium. To see why, consider the strategy profile where all workers apply to f_r repeatedly. Then, if all informed type-θ_h workers continued applying to f_r, each would be hired with probability $\frac{1/4}{1/2} = 1/2$. This cannot be an equilibrium, as type-θ_h workers would deviate and apply to f_s instead. Suppose informed type-θ_h workers would randomize between f_r and f_s such that informed type-θ_h workers are indifferent between the two firms, namely a total of $3/8$ type-θ_h workers apply to f_s so that the expected utility from f_r is 2. However, this would imply that uninformed workers expect a match value less than 2 from applying to f_r, because of the non-zero probability that they are type-θ_l. Uninformed workers would then deviate and apply to f_s. Therefore, all workers applying to f_r indefinitely could not have been an equilibrium.

When δ is sufficiently close to one, all workers will apply at least once to f_r. Since workers are not informed of their type when they are rejected, a portion of the workers that fail their initial application will reapply to f_r. The more workers that reapply, the lower an uninformed worker's expected payoff from f_r will be, due to the competition type-θ_h workers face. As will be shown, when $\delta < 1$, in any equilibrium, uninformed workers with two failed applications to f_r never apply to f_r.

Suppose it was optimal for an uninformed worker to apply more than twice to f_r. Uninformed workers with two failed applications place a lower weight on their probability of being type θ_h than uninformed worker with zero or one failed application. As such, uninformed workers with zero or one failed applications must strictly prefer to reapply to f_r. Then, for δ sufficiently close to 1, almost all workers will have applied at least twice to f_r. Since each type-θ_h worker is hired with a minimum probability of 1/2, a minimum of 3/4 of type-θ_h workers will have been hired at least once at f_r and therefore know their type with certainty.

Informed type-θ_h workers must continue applying to f_r, otherwise applying to f_r while uninformed could not have been profitable. Then, the average rate of hire for type-θ_h workers at f_r will be below 2/3, implying informed workers would prefer to apply to f_s. This proves that any strategy profile where uninformed workers apply to f_r after more than one rejection cannot be an equilibrium. Therefore, all uninformed workers with two failed applications apply to f_s. The reapplication rate will be such that workers with a single failed application are indifferent between reapplying and applying to f_s. In equilibrium, the probability a type-θ_h worker is hired at f_r, denoted by p, and the probability an uninformed worker with a single failed application to f_r reapplies to f_r, denoted by x, must satisfy two equations. First, p must equal $m(f_r)$ divided by the proportion of type-θ_h workers applying to f_r:

$$p = \frac{1/4}{(1-\delta)1/2(1+\frac{\delta}{1-\delta}p+\delta x(1-p)(1+p\frac{\delta}{1-\delta}))}.$$

Let $\gamma_h = \frac{1/2(1-p)}{1/2(1-p)+1/2}$ denote an uninformed worker's posterior probability that she is of type θ_h after a single rejection from f_r. In equilibrium, uninformed workers with a single failed application to f_r must be indifferent between applying to f_r and f_s:

$$\frac{\gamma_h p(3+\frac{\delta 3p}{1-\delta})}{\frac{\gamma_h p}{1-\delta}+(1-\gamma_h p)} = 2.$$

This completes the characterization of equilibrium for Example 3. As δ goes to 1, the proportion of uninformed agents applying to f_r is bounded above, since uninformed agents do not apply to f_r more than twice. However, as δ increases, there are less uninformed agents present each period, and so the mass of agents applying to f_s must increase in turn. In the limit, the equilibrium outcome converges to that of the corresponding market with complete information. The implications of Example 3 can be shown to generalize.

The equilibrium outcomes of aligned markets converge to the corresponding complete information outcomes as $\delta \to 1$. As δ increases, the opportunity cost of applying to firms with uncertain match values decreases. However, when competition is present, the gain from learning one's type also decreases as more workers attempt to learn their type. As $\delta \to 1$, workers who know they are in the class with the maximal match value must attempt to learn if they have access to that match value. In any market, after a finite number of applications to firms, any worker can learn about their type to arbitrary precision. Then, on a type by type basis, as

$\delta \to 1$, outcomes converge to the corresponding outcomes in the complete information case.

A market $\mathcal{M}$ with incomplete preference information can be translated to one with complete preference information. For every worker class c, generate $|c|$ individual classes, in the natural manner. Each type θ belongs to a new class with a degenerate distribution of match values δ_θ and mass $m_c(\theta)$.

Lemma 5 (Equilibrium Outcomes with Long-Lived Workers)**.** *Suppose there are a finite number of worker types, and match values are aligned. For each type-θ, as $\delta \to 1$, the equilibrium probability with which type-θ workers apply to any given firm converges in distribution to the probability class-θ workers apply to that firm in the corresponding complete information market.*[21]

Despite the fact that the long-lived outcome converges to the complete information outcome, there is a marked difference in worker's information in the two economies. For instance, in Example 3, a minimum of $1/4$ type-θ_h workers will never learn their type in equilibrium, for *any* $\delta < 1$. Furthermore, in any equilibrium, a type-θ_h worker that was informed of her type upon arriving would receive strictly higher expected utility than she would if she was uninformed. Both features result directly from oversubscription. For a corresponding complete information market, had the Top-Down algorithm ever encountered oversubscription, workers must apply to firms that are not their most preferred firms with strictly positive probability. This occurs in equilibrium only if workers are not perfectly informed, when $\delta < 1$. Indeed, note that if all workers knew their type, then all would apply to their top choice until the utility from that firm was decreased to that of the next alternative. However, in such an equilibrium in which learning a match value has no benefit, incoming workers would have no incentive to learn their type. Applying to a firm and being rejected, yields a direct loss—not being hired in that given period. By comparison, succeeding fails to change their expected utility, due to the indifference condition. It follows that there must be path dependence in equilibrium for any market with over-subscription.

Lemma 6 (Equilibrium Path Dependence)**.** *In any market with over-subscription, there exists path dependence in equilibrium for any $\delta < 1$.*

Thus even though market outcomes look similar in complete information markets and markets with long-lived workers, the search patterns of workers vary drastically.

[21]Formally, for any $\epsilon > 0$, there exists $\underline{\delta} < 1$ such that for any $\delta > \underline{\delta}$, any type θ, and any firm $j \in \mathcal{F}$, the equilibrium probability a type-θ worker applies to j in any period of the equilibrium with δ is within ϵ of the equilibrium probability that a type-θ worker applies to j in the corresponding complete information market.

4.6 Tenure

On the individual level, the effects of incomplete preference information also naturally capture several of the empirical facts mentioned in the introduction. One consistent trend in the data is that higher earners or those who attended better schools are less likely to state that given the chance they would have changed their major or the school they attend (Network 2017). For instance, while 50% of US adults reported that they would change an important education decision, that number drops to 40% when considering adults at top schools and further drops to 23% when considering adults with incomes above $250,000. Lemma 7 shows that despite having potentially higher outside options, those with higher current matches are more likely to be satisfied with any given match. Let s_θ denote the average stopping age for type-θ workers.

Definition (Stopping Age)**.** *Let $t_s(i)$ be the first period t such that $A_i(t) = A_i(\tau)$ for some $\tau < t$, then $s_\theta \equiv \mathbb{E}_\theta[t_s(i)|i$ is a type-θ worker$]$.*

Lemma 7 (High-Quality Workers Search Less)**.** *Suppose θ_h and θ_l are both types in class c, and θ_h has uniformly higher match values than θ_l; that is, $\theta_h^j \geq \theta_l^j \forall\ f$.*

Then, in equilibrium, $s_{\theta_h} \leq s_{\theta_l}$.

Lemma 7 is a natural consequence of the optimal search policy. Workers apply for the firm with the maximal Gittins index. Since types θ_h and θ_l are members of the same class, c, type-θ_h workers and type-θ_l workers would have identical Gittins indices across firms they have not yet applied to. That is, their option value is identical across firms. However, type-θ_h workers always realize a higher match value than type-θ_l workers. Therefore, if a type-θ_l worker is satisfied with a firm j, then a type-θ_h worker would be satisfied with j as well. Therefore, whenever type-θ_l workers are willing to stop searching at a given firm, type-θ_h workers would also stop searching at that firm. When matches are permanent, this would manifest as type-θ_h workers reporting higher levels of satisfaction with their current matches, in line with the Gallup report.

5 Policy Interventions

I proceed by evaluating the impact of two natural policy interventions designed for markets with incomplete information. I show that congestion alters workers' incentives to search, changing equilibrium outcomes in markets with headhunters and unemployment benefits. While there are many instruments that could be used to increase the quality of matches, I focus on headhunters and unemployment benefits as they are commonly used tools in practice. Furthermore, it is often claimed that these interventions increase workers' propensity to search.

5.1 Headhunters

A natural question is the impact of hidden information, are all workers affected equally by the presence of uncertainty? When firms are congested, incomplete preference information benefits workers with high match values. Under incomplete information, workers must apply to multiple firms to find their best matches. This search comes at the cost of time and firings for workers with low match values. Effectively, the ability of workers with low match values to apply to other firms is restricted, leaving more spots open to other workers. Furthermore, even when preferences are aligned, firms' incentives may be in line with certain workers' incentives but not others.

Suppose firm f_r hired a headhunter to target workers with high match values at firm f_r. The headhunter then could send out offers to desirable workers, informing them of their match values at f_r, $(\theta_w^{f_r}, \theta_f^{f_r})$. Workers that were not approached would be able to infer that they had a low match value at firm f_r. To formalize this extension, I define the market with *revelation at firm* j as the market induced by all workers learning their match value at firm j.

Definition (Revelation at Firm j)**.** *Let market* $\mathcal{M} = (\mathcal{F}, \mathcal{C}, m)$ *be given.*

Then, market $\mathcal{M}_j = (\mathcal{F}', \mathcal{C}', m')$ *is induced by revelation at firm* j *if:*

1. $\mathcal{F}' = \mathcal{F}$,
2. $\mathcal{C}' = \cup_{c \in \mathcal{C}} \left\{ \cup_{x \in \mathbb{R}^+} \{\theta | ((\theta_w^j, \theta_f^j), \theta_{-j}) = \theta \in c\} \right\}$,
3. $m'_{c'}(\theta') = m_c(\theta')$, $m'(f) = m(f)$.

Condition 2 splits each worker class c into k_c worker classes, where each new class involves a unique match value at firm j. Condition 3 simply requires that the mass function is proportionally distributed across the new classes.

Returning to Example 2, in the market with revelation at firm f_r, the equilibrium is simple: all type-θ_h workers apply to f_r, while class-c_2 workers randomize between the two firms and type-θ_l workers apply to f_s. Firm f_r benefits from this change in outcome, as it receives higher profit than in the original equilibrium. However, revealing information does not always benefit workers with high match values at that firm. Example 4 illustrates how headhunters can benefit workers with low match values to the detriment of other workers.

Example 4 (Revelation)**.**

Workers \ *Firms*	$m(f_r) = 1/4$	$m(f_s) = 1/2$
$m(\theta_h) = 1/2$	$(3, 3)$	$(2, 2)$
$m(\theta_l) = 1/2$	$(1/2, 1/2)$	$(2, 2)$

Revealing information at f_r benefits type-θ_l workers through inducing them to avoid f_r. Type-θ_h workers are harmed through increasing competition at f_s.

When $\delta = 1/2$, equilibrium payoffs for workers of type θ_h and θ_l are $\frac{9}{4}$ and $\frac{5}{4}$, respectively. The equilibrium strategy profile is simple. Workers randomize between the two firms when uninformed, and apply to f_r if type θ_h or to f_s if rejected initially from f_r. In equilibrium, when type-θ_l workers are rejected after applying to f_r, the mass of workers that can apply to f_s is reduced. The low capacity of f_r requires type-θ_h workers to apply to f_s in equilibrium, and so they benefit from the rejections suffered by type-θ_l workers.

Now consider $\mathcal{M}_r$. Type-θ_l workers know they will never be hired by f_r. As such, all type-θ_l workers apply to f_s. In doing so, they decrease the hiring probability at f_s. This weakens the outside option for type-θ_h workers who previously benefited from access to f_s, but now have a reduced safety valve for congestion at f_r. In market $\mathcal{M}_r$, both types' equilibrium payoffs are $\frac{7}{4}$. Type-θ_l workers are made better off by the revelation, while type-θ_h workers are harmed. In this stylized example, complete information harms type-θ_h workers relative to type-θ_l workers because the information itself provides no aggregate increase in efficiency.

In general, firm j hiring a headhunter has two effects; a sorting effect, where workers with high match values at j can immediately apply to j, and a congestion effect, where workers with low match values at j apply to safer options. When j is sufficiently congested, the former effect vanishes, leaving only the latter, benefiting workers with low match values at j on the whole. If j is not sufficiently congested, the result of this policy is ambiguous. Proposition 5 characterizes these two effects, and provides a sufficient condition for less competitive types to be better off than more competitive types.

Two definitions will be useful. First, a market is *congested* if each firm fully utilizes its capacity in equilibrium.

Definition (Congestion)**.** *Firm j is congested, if in equilibrium, $D^j = m(j)$. The market is congested if every firm is congested.*

Second, to economize on notation, let $U^j_{\mathcal{M}}(\theta)$ denote type-θ workers' payoff from firm j in market $\mathcal{M}$:

$$U^j_{\mathcal{M}}(\theta) = \mathbb{E}_{\mathcal{M}}\left[\sum_{t=0}^{\infty} \delta^t \mathbf{1}_{w(t)=j}\theta^j_w\right]$$

Proposition 5 (Impact of Headhunters)**.** *Suppose $\mathcal{M}$ is congested and θ_h and θ_l are members of the same class, but $\theta^j_h > \theta^j_l$.*

1. *The increase in type-θ_h workers' matches with firm j is more than type-θ_l worker's increase:*

$$U^j_{\mathcal{M}_j}(\theta_h) - U^j_{\mathcal{M}}(\theta_h) \geq U^j_{\mathcal{M}_j}(\theta_l) - U^j_{\mathcal{M}}(\theta_l).$$

2. *If $m(\theta^j_h) > m(j)$ and $|C| = 1$, type-θ_h is harmed relative to type-θ_l:*

$$U_{\mathcal{M}_j}(\theta_h) - U_{\mathcal{M}}(\theta_h) \leq U_{\mathcal{M}_j}(\theta_l) - U_{\mathcal{M}}(\theta_l).$$

Part 1 of Proposition 5 shows that workers with high match values at firm j receive higher net utility from firm j. Part 2, whose condition was satisfied in Example 4, states that on net, a worker with high match values at firm j gains less from headhunters when j is heavily congested. There are two forces behind the second result: the nature of the outside option and the independence of match values across firms. In decentralized markets, the cost of a poor match is embedded in the opportunity cost. A centralized market, wherein a rejected worker could immediately apply to another firm within the same period, would eliminate this force. In such a setting, this result would reverse, headhunters would solely benefit workers with high match values at j. Similarly, if types were correlated across firms, that is if $\theta^j_h > \theta^j_l$ implied $\theta^{j'}_h > \theta^{j'}_l$, then the competition effect would vanish. This follows because θ_h would not be effected by the increased level of congestion at f_s.

5.2 Unemployment Benefits in Congested Markets

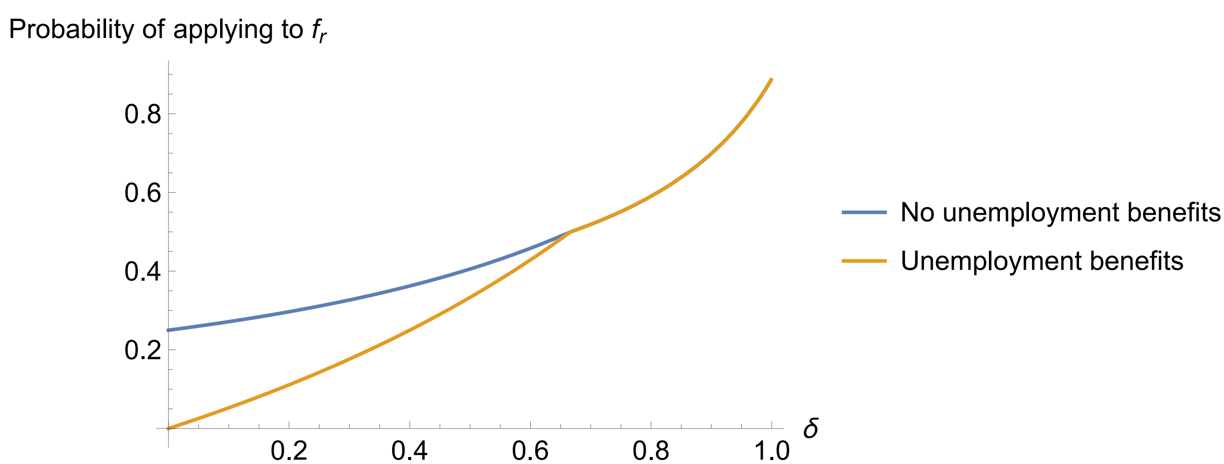
A common intuition holds that increasing unemployment benefits further encourages search, because it reduces the loss from rejection. In turn, unemployment benefits could be expected to improve equilibrium efficiency as workers and firms are better matched. $\theta^{\varnothing}_w$ has a straightforward interpretation in this context, $\theta^{\varnothing}_w$ is the benefit for rejected workers, or the "unemployment benefits." This section considers the impact when the utility from being rejected in a given period is positive. Example 5 shows that unemployment benefits can have a nuanced effect on equilibrium efficiency. *Negative learning*, a worker's incentive to learn about a less desirable firm, is suppressed by unemployment benefits.

Example 5 (Unemployment Benefits)**.**

Firms / *Workers*	$m(f_r) = 1/4$	$m(f_s) = 1/4$
$m(\theta_h) = 1/2$	$(3, 3)$	$(3, 3)$
$m(\theta_l) = 1/2$	$(1, 1)$	$(3, 3)$

As depicted in Figure S1, increasing unemployment benefits when there is an agreed upon top firm can decrease equilibrium welfare.

FIGURE S1: The x-axis represents δ, the probability a worker does not retire. The blue line shows the proportion of uninformed workers that apply to f_r when $\theta_w^{\varnothing} = 0$. The yellow line shows the proportion of uninformed workers that apply to f_r when $\theta_w^{\varnothing} = 1$.



Safety nets can reduce the overall efficiency of equilibrium. Indeed, when the rate of applications for uninformed workers to f_r drops below 1/4 in Example 5, the total equilibrium utility, net of $\theta_w^{\varnothing}$, is reduced. The key feature in Example 5 is that f_s is commonly known to be more attractive than f_r. Prior to the introduction of unemployment benefits, workers refrain from applying to f_s due to the high level of competition and resulting low probability of acceptance. As $\theta_w^{\varnothing}$ increases, the loss from rejection decreases. The lowered loss incentivizes more workers to apply to f_s, in order to equilibriate the probability of rejection. However, this reduces equilibrium efficiency when δ is low. The increase in applicants to f_s fails to improve matching, while too few type-θ_h workers are hired by f_r.

6 Transferable Utility

I show transferable utility can be simply incorporated into the transient matching model. Furthermore, doing so does not cause worker demand to violate gross substitutes, and therefore does not change the qualitative results. I consider a competitive equilibrium environment where firms strategically choose wages, before workers apply to firms.

6.1 Competitive Equilibrium

In a competitive equilibrium, firms choose wages based on observables—match values and classes—to maximize profits. Formally, I consider a separate "wage game," which occurs before the market resolves in period 0. In the wage game, all firms simultaneously announce a *payment function* $\phi_j : \mathcal{C} \times \mathbb{R}_+ \to \mathbb{R}$, denoting the payment from firm j to a worker conditional on that worker's class, c, and firm match value, θ_f^j. Importantly, workers of a given class and equal match values must receive the same payment, even if their type differs.

To maintain the connection with the previous sections, I assume the prenumeration match values remain unchanged. If a class-c type-θ worker matches to firm j, the two agents receive θ_w^j and θ_f^j, respectively, in addition to the transfer. That is, they receive $\theta_w^j + \phi_j(c, \theta)$ and $\theta_f^j - \phi_j(c, \theta)$. After period 0, wages remain fixed, and workers apply to firms as before, with match values altered accordingly.

Importantly, firm j evaluates his profit through his per-period expected equilibrium profit in the market game, denoted by $\pi_j(\{\phi_f\}_{f \in \mathcal{F}})$. This is in contrast to the previous sections, where firms evaluated profits through their discounted stream. To see why this is necessary, recall the previous discussion of alignment. Alignment was used to rule out the possibility of rejection cycles, wherein a firm might reject one worker in order to receive an application from a preferred worker. Here, with arbitrary transferable utility, there is no assurance that the resulting market will be aligned. To retain the focus on workers' search strategies, firms are required to be myopic in evaluating utility.

I assume firms are myopic—when hiring, firms maximize profit in the current period—and firms randomize when indifferent. This rules out firms incentivizing more or less experimentation through selectively hiring certain worker classes. When each firm is made up of many small teams, each of which must make a hiring decision, this assumption is natural. Each individual team has a small impact on the overall matching and so focuses on maximizing their own utility. Alternatively, if each "firm" were a unique industry comprised of many firms, similar logic would imply that each firm optimizes by maximizing the current period utility.

Assumption (Myopic Firms)**.** *Firms maximize current period profit. Furthermore, if a firm is indifferent among a subset of its applicants, and cannot hire the entire subset, it uniformly* *randomizes over that subset.*

Wages are required to satisfy a limited liability condition—the size of any transfer cannot exceed the gain from the matching for either party:

Assumption (Limited Liability)**.**

$$-\theta_w^j \leq \phi_j(\theta) \leq \theta_f^j$$

I proceed by defining *competitive equilibrium* in the standard manner. A profile of payment functions constitutes a competitive equilibrium if no firm can change his payment function to increase his profit:

Definition (Competitive Equilibrium)**.** *A profile of payment functions* $\{\phi_j\}_{j\in\mathcal{F}}$ *constitutes a competitive equilibrium if for all* j, ϕ*:*

$$\pi_j\left(\{\phi_k\}_{k\in\mathcal{F}}\right) \geq \pi_j\left(\phi, \{\phi_{j'}\}_{j'\neq j}\right)$$

This equilibrium characterization implicitly prohibits dynamic punishments. Allowing for dynamic punishments would enable a folk-theorem style argument which could rationalize all wages when firms are sufficiently patient.

Importantly, for any profile of payment functions, the resulting market is effectively a new market with non-transferable utility. Then, workers still evaluate firms as if they were endogenous bandits, albeit with differing rewards. This implies that aggregate demand satisfies gross substitutes, and therefore an equilibrium can be characterized using the algorithm in Theorem 1.

Proposition 6 (Wages Satisfies Gross Substitutes)**.** *For any profile of wages* $\{\phi_j\}_{j\in\mathcal{F}}$*, aggregate demand satisfies gross substitutes.*

6.2 Non-Existence of Competitive Equilibrium

Even in such a simple environment, existence of competitive equilibria is not guaranteed. Two issues arise: a type's demand for a firm is not necessarily continuous in wage, as workers can apply multiple times; and firms cannot condition wages on worker's outside options, as they do not observe worker's types.

Example 6 (Competitive Equilibria Need Not Exist)**.**

Firms / *Workers*	$m(f_r) = 1/2$	$m(f_s) = 2$
$m(\theta_h) = 1$	$(3, 3)$	$(2, 2)$
$m(\theta_l) = 1$	$(1/2, 1/2)$	$(2, 2)$

First, note that the minimum proportion of workers f_s can hire is given by the strategy profile where all workers initially apply to f_r, then type-θ_l workers subsequently apply to f_s. Note that if a type-θ_l worker is willing to remain at f_r after learning her type, then all type-θ_h workers must strictly prefer to remain at f_r. Furthermore, this implies that an uninformed worker must strictly prefer to work at f_r. However, then the mass of workers applying to f_r is greater than $1/2$, so type-θ_l workers are hired with probability 0. Type-θ_l workers would then prefer to apply to f_s, unless f_s's wage is exactly -2, in which case type-θ_l workers are indifferent. By standard bargaining arguments, this implies that $\phi_s = -2$ and all type-θ_l workers apply to f_s. Then, a lower bound for π_s in equilibrium is given by $1/2\delta \cdot 4 = 2\delta$.

To show no competitive equilibrium exists in Example 6, suppose for contradiction a competitive equilibrium exists. There are three possible categories of strategies for workers. 1) All workers apply to f_s forever; 2) all workers initially apply to f_r, then type-θ_h workers remain at f_r and type-θ_l workers migrate to f_s forever; and 3) workers randomize between the previous two options.

First, suppose in the competitive equilibrium, all workers applied to f_s forever. Then, f_r would hire no workers, and so receive zero profit. f_r would deviate and set payment for θ_h equal to $6 - \epsilon$, since this would generate positive profit for f_r. However, f_s would never offer a wage higher than $2\delta = (2 - w)2$ or $w = 2(1 - \delta/2)$ in equilibrium, as otherwise f_s could revert to a wage of -2. Since $2(1 - \delta/2) < 6$ at least one of f_r or f_s must have a profitable deviation.

Next, suppose all workers initially apply to f_r. Then, f_s must set its wage to -2 as listed above. However, if f_r's wage was above -3, f_r could deviate to a wage of $-3 + \epsilon$ while still attracting type-θ_h workers. This implies f_s could set a wage just above ϕ_r, a marginal increase in payment that earns f_s a strict increase in workers of $1 - 1/2\delta$.

Last, workers randomized between the two firms. Then, workers are exactly indifferent between the two firms. However, a firm could increase its wage marginally to capture all workers. This generates a strict increase in profit, unless that firm was earning zero profit from each worker. For f_r this implies $\phi_r = 3$, while for f_s this implies $\phi_s = 2$. However, then f_s could deviate to $\phi_s = -2$ as detailed above.

The driving force behind the non-existence argument is that firms cannot distinguish workers with different outside options. This reduction in the dimension of possible wages restricts the ability to equilibriate demand and supply.[22]

6.3 Resumes

The non-existence in the previous example can be resolved by the inclusion of resumes. In this context, a resume allows a worker to prove she had been previously hired by a separate firm, in return for higher wages. Resumes allow for wages to be conditioned not only upon the direct match value, but also upon a worker's previous history. Formally, a *resume-dependent payment function* is a payment function defined on a larger domain $\phi_j : \mathcal{C} \times \mathbb{R}_+ \times \{A_i^j(\tau)\}_{\tau=0}^t \to \mathbb{R}$.

To see how the inclusion of resumes generates a competitive equilibrium in Example 6, note that now f_s could choose a wage that incorporates the worker's outside option. The following wages form equilibrium for sufficiently high δ: type-θ_l workers receive 0 from f_r and -2 from f_s, while type-θ_h workers receive 1 from f_r and 2 from f_s upon providing a resume and -2 otherwise. In equilibrium, all workers apply to f_r initially, and type-θ_l workers transition to working at f_s.

The equilibrium non-existence arose due to the requirement that firms treat workers of a single class with equal match values identically. Through introducing resumes, workers could prove that their outside options were distinct. Thus, resumes play an important role in information transmission.

7 Discussion

This paper develops a framework for analyzing transient matching when workers learn through experimentation. Firm capacity constraints force workers to anticipate other workers' application decisions. I show that, firms can be evaluated as endogenous bandits. Once firms' hiring decisions are described as thresholds, techniques from the multi-armed bandit literature allow for a simple description of the optimal worker policy. Importantly, aggregate demand satisfies the gross substitutes condition, which enables the characterization of equilibrium. Workers' search patterns match data from labor markets, high-quality workers report higher satisfaction—despite not having better information.

I show that the nature of both learning and competition are critical to understanding the impact of policy interventions. Commonly used interventions, such as hiring headhunters or

[22]It is worth noting that while on the surface this may appear similar to issues such as the non-existence that arises from adverse selection in Rothschild and Stiglitz (1978), the non-existence here comes from a different source. In their work, non-existence results from free firm entry, as firms currently in the market are harmed by entry. In transient markets, the firms already present in the market face incentives to change their pricing structures, in order to manipulate the learning behavior of workers in the market.

increasing unemployment benefits, may generate unintended effects in congested markets. For instance, headhunters may redistribute the benefits from matching, while increasing unemployment benefits can intensify competition.

The results also imply careful consideration should be taken before changing centralized mechanisms. Such mechanisms often feature decentralized aftermarkets, where the incoming information from the original centralized market can radically shift the final outcome. This paper provides a first step towards better understanding the effects changes of mechanism rules can have on the aftermarkets of centralized mechanisms.

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A Proofs

I begin by showing that standard results from the operations literature apply when firms hire all applicants. Lemma 8 provides the basis for the index-related arguments in the rest of the appendix.

Lemma 8. *Let σ be an equilibrium, such that, in any subgame, all firms hire all applicants each period. In any period, each worker i applies to some $j \in \arg\max_j GI_i^\sigma(j, h_t^i)$.*

Proof of Lemma 8

This proof follows directly from Theorem 2.1 of (Gittins, Glazebrook, and Weber 2011) which states:

Theorem 2.1: *A policy for a simple family of alternative bandit processes is optimal if it is an index policy with respect to the Gittins index of each bandit process.*

Then, it must be shown that the set of firms acts as a simple family of alternative bandit processes for each worker. First, since firms hire all applicants, each worker i faces a fixed decision problem for any strategy profile σ, independent of the other workers' strategies. For any possible worker strategy profile, each worker is always hired, no matter which firm she applies to. As such, the payoff for a type-θ worker from applying to firm j, is simply θ_w^j in any equilibrium.

Each firm is then a bandit process that can either be activated or frozen. A frozen firm provides no payoff, while an activated firm provides θ_w^j. Each worker must activate exactly one firm each period. The independence of firm match values—conditional on a worker's class—implies that the set of firms is a simple family of alternative bandit processes. Theorem 2.1 from Gittins, Glazebrook, and Weber (ibid.) then applies, and the result follows. ■

Proof of Lemma 1

I begin by describing the Top-Down algorithm in detail. For any complete-information market $\mathcal{M}_I = (\mathcal{F}, \mathcal{C}, m)$, the Top-Down algorithm proceeds as follows:

Top-Down Algorithm:

Let $\mathcal{E} = \mathcal{F} \cup \mathcal{C}$ be the set of agents present in the market.

1. Find the largest worker-firm match value: $(c^*, j^*) = \arg\max_{c,j\in\mathcal{E}} \theta_c^j$, the finiteness of $\mathcal{C}$ implies this is well-defined.

2. Find class-c^*'s second-highest match value: $j_2 = \arg\max_{j\neq j^*} \theta_{c^*}^j$.

3. Match c^* to j^*, which leads to one of two possible outcomes:

(a) Undersubscription—When all class-c^* workers apply to j^*, j^* is still preferred by class-c^* workers relative to j_2:

$$\frac{m(j^*)}{m(c^*)}\theta_{c^*}^{j^*} \geq \theta_{c^*}^{j_2}$$

i. All class-c^* workers apply to j^* forever.

ii. Firm j^*'s capacity is reduced by the mass of applicants: $m(j^*) = \max\{0, m(j^*) - m(c^*)$.

iii. Reduce to the submarket without class-c^*, $\mathcal{E} = \mathcal{E} \setminus \{c^*\}$. Remove j^* from the market as well if $m(j^*) = 0$.

(b) Oversubscription—When too many class-c^* workers apply to j^*, j_2 becomes preferable:

$$\frac{m(j^*)}{m(c^*)}\theta_{c^*}^{j^*} < \theta_{c^*}^{j_2}$$

i. c^* randomizes between j^* and j_2 to generate indifference: c^* applies to j^* with probability $m(j^*)\frac{\theta_{i^*}^{j^*}}{\theta_{c^*}^{j_2}}$. The remaining probability, $1 - m(j^*)\frac{\theta_{i^*}^{j^*}}{\theta_{c^*}^{j_2}}$, will be accounted for in a future step.

ii. Reduce to the submarket without j^*, $\mathcal{E} = \mathcal{E} \setminus \{j^*\}$.

4. In the new submarket $\mathcal{E}$, again select the highest match value: $(c^*, j^*) = \arg\max_{c,j\in\mathcal{E}} \theta_c^j$.

(a) If j^* was not previously selected, repeat steps 2 and 3.

(b) Otherwise, if c^* was previously oversubscribed:

i. The proportion of c^* applying to the original firm must be increased to keep c^*'s payoffs from both firms equal.

ii. Redistribute workers between the two (or more) firms keeping c^*'s payoff from each firm equal, continuing until all of c^* has been allocated, or the payoffs from each firm are equal to those of the next top choice of c^*.

(c) If j^* was previously selected as j_2, increase the proportion of the previous class-c^* applying to the previous j^* to equalize the previous c^*'s utility between the previous j^* and the current j^*.

5. Repeat the above steps until either $\mathcal{E} \subset \mathcal{C}$ or $\mathcal{E} \subset \mathcal{F}$

6. Any remaining workers apply to any firm and are rejected. All remaining firms operate below capacity.

Now I prove the claim. Suppose there exists a strategy profile σ' whose outcome does not coincide with the Top-Down algorithm. The Top-Down can be used to construct a strategy profile σ. Because σ' and σ differ in their outcome, there is a minimum k such that in the kth iteration, there exists a class c, such that c's strategy profile differs between σ and σ'. By construction, this implies that some member of class c has payoff under σ' below its payoff under σ. Furthermore, all matches corresponding to higher match values are equivalent between σ and σ' by construction. Then, under σ', a class c worker could deviate to their firm under σ and receive higher expected utility every period. Therefore, σ' could not have been an equilibrium. ■

Proof of Proposition 1

Consider a firm j that is hiring below capacity, namely j rejects applicants while $m(A_j) < m(j)$. Importantly, rejecting some such applicant, i, has two impacts that could potentially benefit j: the rejection could cause i to re-apply to j in a later period, and the rejection could trigger a rejection cycle wherein a more preferable worker i' is rejected by another firm causing i' to apply to j.

Hiring i immediately front-loads the match value from i, avoiding the loss from i possibly retiring before returning to j. Note that it cannot be the case that i eventually returns to j and applies more times to j than i would if j had accepted immediately. Eventually i's information sets under the original strategy and the deviation must coincide, at which point the Markovian nature of i's strategy forces i to apply to the same firm under both strategy profiles. Then, it follows that j's payoff from that point forward is unchanged. Since i was accepted by j at most once before i's information converged under the original strategy profile, j strictly benefits from the deviation.

By assumption, match values are aligned. It is known that aligned markets have no rejection cycles (Voorneveld and Norde 1997). Namely, it cannot be the case that $\theta_i^{j'} > \theta_{i'}^{j'}, \theta_i^j > \theta_i^{j'}$, and $\theta_{i'}^{j'} > \theta_{i'}^j$. Then, j can not benefit from rejecting i in an attempt to attract other workers. Since j cannot benefit regardless from rejecting i, j must accept as many applicants as possible in equilibrium.

Last, if j is congested, then it must have excess applications with match values equal to its lowest match value. In order to accept a high ranked applicant, j would then need to reject one of its lowest ranked applicant. However, even rejecting that lowest ranked applicant dissuades her from applying in future periods, another worker with equivalent match value will be available to replace it. Then, the previous arguments imply that j has a profitable deviation.

■

Proof of Lemma 2

A Markovian strategy for firm j, is a mapping from the types of workers, a_j, to acceptances.

Notably, because worker's strategies are Markovian, worker's payoff-relevant information is summarized by her belief regarding her type. For a given worker j, and private history h_t^j, let j's posterior regarding her type be given by $p \in \Delta\Theta$. Then, the payoff-relevant state space is a distribution over p, Ψ.

Consider the Markov chain over Ψ. Workers' applications follow the same initial distribution over applications. Furthermore, there are a finite number of firms, and as such workers must eventually converge in belief, independent of the order in which they apply to firms, implying that their final applications are a fixed quantity. Then, there exists a subset $V \subset \Psi$, such that V is irreducible. Furthermore, because new workers enter the market every period, the Markov chain is aperiodic. Then, by the Steady State Theorem, there exists a steady state. ■

Proof of Lemma 3

Suppose for worker i, under an arbitrary strategy profile σ, for some firm j, $GI_i^\sigma(j, h_t^i) = x$. I show that this value of $GI_i^\sigma(j, h_t^i)$ is achieved by the strategy described in the statement of the lemma: if the realization of ϕ_i^j is above x set $\tau = \infty$, otherwise set $\tau = 1$. If exactly x was realized, any stopping time yields an equivalent outcome. Assume throughout that the realization is given by $y > x$. It will be helpful to refer to the value of the stopping time problem that characterizes $GI_i^\sigma(j, h_t^i)$. I let $gi(f, h_t^i, \tau)$ be the value of the stopping time problem that characterizes $GI_i^\sigma(j, h_t^i)$, given a possibly suboptimal stopping time, τ. Formally:

$$gi(f, h_t^i, \tau) \equiv \frac{\mathbb{E}_i\left[\sum_{t=1}^{\tau} \delta^t \phi_i^j\right]}{\mathbb{E}\left[\sum_{t=1}^{\tau} \delta^t\right]}$$

I use induction to show that, for any stopping time $\tau < \infty$, $gi(f, h_t^i, \tau + 1) > gi(f, h_t^i, \tau)$.

To begin, I consider the case where $\tau = 1$:

Let $x = p/q$ where $q > 0$, p and q are not necessarily integers, no rationality assumption is made.

$$\begin{aligned}
\frac{p + \delta y}{q + \delta} &> p/q \\
p + \delta y &> \frac{p(q + \delta)}{q} \\
\delta q y &> \delta p \\
q y &> p \\
y &> p/q = x
\end{aligned}$$

Next, suppose that the claim holds for $\tau \in \{1, 2, \ldots, k\}$. A similar computation implies the inductive step also holds for $\tau + 1$.

$$\frac{p+\delta^{k+1}y}{q+\delta^{k+1}} > p/q$$
$$\implies y > p/q = x$$

Then, for any k, $\frac{p+\delta^k y}{q+\delta^k} > p/q$, therefore $gi^{\sigma}(f, h_t^i, \tau) = GI_i^{\sigma}(j, h_t^i)$ only if $\tau \in \{1, \infty\}$.

Conversely, the argument also shows that when $y < x$, setting $\tau = 1$ is optimal.

Then, the claim is proven, $GI_i^{\sigma}(j, h_t^i)$ is characterized by the strategy described in the statement of the lemma. ■

Before proving Proposition 2, I prove a useful lemma. Strict dynamic preferences implies that two equilibrium strategy profiles that induce different outcomes must also generate different hiring thresholds.

Lemma 9. *Under strict dynamic preferences, if σ and σ' generate distinct equilibrium outcomes, then there must exist some firm j whose hiring threshold under σ differs from his hiring threshold under σ'.*

Proof of Lemma 9

Suppose not. That is, σ and σ' induce different outcomes, but every firm has identical thresholds under σ and σ'. Then, every worker faces identical Gittins indices under σ and σ', at every firm, under any informational partition. In particular, since σ and σ' have distinct outcomes, there exists a type θ and history h_t^i such that type-θ workers with history h_t^i make different choices under σ and σ', and type-θ workers are hired with positive probability. Without loss of generality, suppose type-θ workers apply for firm j with greater probability under σ than under σ'. Similarly, type-θ workers must apply to another firm j' with greater probability under σ' than σ. Since forward induction policies are optimal, this implies that $GI_i^{\sigma}(j) = GI_i^{\sigma'}(j')$. Then, strict dynamic preferences implies that type-θ workers must be hired with intermediate probability at either firm j or firm j'. Namely, the threshold for either j or j' is set at a match value of θ_f^j or $\theta_f^{j'}$.

First, suppose type-θ workers were hired with intermediate probability at j. As type-θ workers leave j, if no other workers start applying to j, the probability of being hired increases for remaining type-θ workers due to the reduced competition, thereby increasing $GI_{\theta}^{\sigma'}(j)$ contradicting the claim. To prevent this, there must be another type θ', whose workers apply to firm j in increased numbers. However, type-θ' workers willingness to do so implies that there exists another firm j'' such that $GI_{\theta'}^{\sigma}(j) = GI_{\theta'}^{\sigma}(j'')$. Again, by assumption 1 workers of this type must be hired with intermediate probability at one of these firms. Since type-θ workers were hired with intermediate probability at firm j, type-θ' workers must be hired with intermediate probability at firm j''. Repeating this line of logic implies that there exists a cycle of workers, each facing equal Gittins indices at least two firms and hired with intermediate probability at one such firm. However, inherently the total mass of workers is fixed, for every mass of workers that leave a firm, an equal mass must take their place. But marginal workers have a probability less than 1 of being hired, while those taking their place do not. Then, such a cycle cannot keep the total amount of workers hired at involved firms equal. Therefore, the thresholds at those firms must either increase or decrease, contradicting the original assumption. ∎

Proof of Proposition 2

Suppose σ and σ' are both equilibria, with distinct outcomes. Lemma 9 then implies that there exists some firm j such that firm j's threshold under σ is distinct from its threshold under σ'.

Without loss of generality, suppose firm j's threshold is higher under σ than under σ'. Then, consider worker i, where i previously applied to a firm j' but now applies to j. A positive mass of such workers must exist, otherwise j's threshold could not have increased. One of two cases must have occurred to generate this change in application, either worker i's original firm's Gittins index decreased, $GI_i^{\sigma'}(j', h_t^i) < GI_i^{\sigma}(j', h_t^i)$, or worker i's Gittin index at their new firm increased, $GI_i^{\sigma'}(j, h_t^i) > GI_i^{\sigma}(j, h_t^i)$. However, $GI_i^{\sigma'}(j, h_t^i)$ cannot be larger than $GI_i^{\sigma}(j, h_t^i)$ because firm j's threshold has increased, therefore every worker has a weakly lower firm j Gittins index than before. Then, $GI_i^{\sigma'}(j', h_t^i) < GI_i^{\sigma}(j', h_t^i)$, implying that the threshold at firm j' has increased.

Then, the same line of logic as before implies that there exists another non-zero mass of workers applying to firm j', and their original firm must have increased its hiring thresholds. Since there are a finite number of firms, eventually this chain must return to a firm it has previously visited, generating a set of firms with higher thresholds under σ' relative to σ.

Consider the implications of such a set, it cannot be the case that a smaller total of workers are being hired at these firms under σ, otherwise thresholds would fall accordingly. Furthermore, since thresholds are higher, the total utility of hired workers must be higher under σ relative to σ'. Then, there must be at least one worker with strictly greater utility than before. Then, such a worker could have profitably deviated in σ' to its strategy under σ. However, this contradicts the fact that σ' was an equilibrium. ∎

Proof of Lemma 4

Lemma 8 showed that in equilibrium workers must apply to a firm with maximal Gittins index when capacities were unconstrained. Repeating the proof of Lemma 8, updating the rewards from firm j, replacing θ_w^j with $\psi_\theta^j(v,p)$ immediately implies that the set of capacity constrained firms is still a simple family of multi-armed bandits. The result follows. ■

Proof of Proposition 3

I begin by proving a stronger claim for individual demand. I show that when whenever any type-θ worker faces a multi-armed bandit problem, and type-θ worker's Gittins index for any firm j, $GI_\theta(j)$, is decreasing in j's threshold, (v_j, p_j), type-θ's demand will be decreasing in j's threshold as well. Gross substitutes is satisfied in multi-armed bandit problems because the Gittins index for each individual firm is only a function of the rewards from that firm. Then, increasing the threshold of firm j_2 cannot affect the Gittins index of firm j_1. Furthermore, the realized demand over each firm is weakly increasing in that firm's Gittins index. As such, raising the thresholds of a set of firms, decreases the Gittins indices of those firms, but fails to change the Gittins indices of other firms. Then, applications to the original set of firms must weakly decrease.

Theorem 2 (Bandits imply Gross Substitutes)**.** *Suppose for all θ, type-θ workers face a simple multi-armed bandit problem, and for any firm j, $GI_\theta(j, h_t^i)$ is decreasing in (v_j, p_j). Then, type-θ's demand satisfies gross substitutes.*

Proof of Theorem 2

Lemma 8 implies that, when firms hire all workers, workers apply to a firm with maximal Gittins index each period. However, for any firm j and threshold (v_j, p_j), an auxiliary firm j' can be defined, where type-θ's match values are given by $\psi_\theta(v_j, p_j)$. Then, j' hires all workers, and so Lemma 8 applies.

Then, let $\mathbf{F}$ be a set of firms, $\mathbf{F} \subset \mathcal{F}$, and $(v,p), (v',p')$ be two vectors of thresholds, such that $(v_j, p_j) = (v_j', p_j')$ for $j \in \mathbf{F}$ and $(v_j, p_j) \geq (v_j', p_j')$ otherwise. Because the thresholds for firms in $\mathbf{F}$ remain unchanged, their Gittins indices are also equal under (v,p) and (v',p'). However, the Gittins indices for firms outside of $\mathbf{F}$, must be weakly lower than before under (v,p). Then, the total demand for firms in $\mathbf{F}$ must weakly increase, as the corresponding Gittins indices are always higher in relative terms.

■

Returning to the proof of Proposition 3, it must be shown that for a given set of thresholds, aggregating across workers preserves changes in demand. This immediately follows from monotonicity of integration. By the definition of aggregate demand, for a given firm $j \in \mathbf{F}$, $D_j(v,p) = \int_\theta D_j^\theta(v,p)d\theta$. Then, if for all θ, $D_j^\theta(v,p) \geq D_j^\theta(v',p')$, it must be that $D_j(v,p) \geq D_j(v',p')$. As such, when every individual type's demand satisfies gross substitutes, so must aggregate demand. The result follows. ■

Before addressing Theorem 1, I prove the following helpful lemma:

Lemma 10. *Suppose after some private history, h_t^θ, which occurs with positive probability for type-θ workers; type-θ applies to j with positive probability, $v_j = \theta^j$, and $GI_\theta(j, h_t^\theta) = GI_\theta(j', h_t^\theta)$. Then, a marginal decrease in p_j yields a discontinuous decrease in D^j.*

Proof of Lemma 10

Forward induction implies a form of independence of irrelevant alternatives, reducing $GI_\theta(j)$ below $GI_\theta(j')$ only affects type-θ workers' choices regarding whether to apply to j or j'. In particular, there is exactly one change, type-θ workers that would have applied to j apply to j' instead. Afterwards, type-θ workers will either remain at j' or apply next to j.

Even if type-θ workers later return to j, a period by period comparison shows that type-θ workers' demand for j has decreased. In the first period, the number applying to j is zero, since all have applied to j'. Then, type-θ workers either remain at j' forever, in which case the claim follows, or they proceed by applying to j. However, even should type-θ workers apply to j, upon having applied to both firms, the impact of decreasing $GI_\theta(j)$ has washed out. Type-θ workers now have applied to both firms. Subsequently, type-θ workers will be weakly more likely to apply to firm j'. Last, since $\delta < 1$ front loading applications to firm j' decreases the total number of type-θ workers applying to firm j. ■

Proof of Theorem 1

I begin by describing the threshold adjustment process in more detail:

Threshold adjustment process:

1. Begin by setting all thresholds to $(v_j, p_j) = (0, 1)$.

2. Determine the aggregate demand vector, D.

3. If no firms are hiring beyond their capacities; $\forall j, D_j \leq m(j)$, terminate.

4. Otherwise, take an arbitrary firm j that is over capacity, and impose its capacity continuously:

   (a) Select the worst match quality θ^j for firm j that is hired with positive probability. Let $\underline{v}_j = \min_{\{\theta|\theta^j \geq v_j\}} \theta^j$ and let c be the class of $\theta \in \arg\min_{\{\theta|\theta \geq v_j\}} \theta^j$.

   (b) Continuously decrease θ's hire probability at j, $p_j \to 0$.

   (c) Three events could result:

      i. Lowering p_j equates demand and capacity, $D^j = m(j)$, continue to step 5.

      ii. Type-$\underline{v}_j$ workers are never hired, $p_j = 0$, return to step 4a and repeat.

iii. Lowering p_j equalizes c's Gittins indices at j and at least one another firm j', $GI_i(j) = GI_i(j')$. Increase type-θ workers' demand for j' at this information set while lowering their demand for j until the demand for j is 0 or $D^j = m(j)$. Proceed to step 4a or 5 accordingly.

5. Again select a firm j such that $D^j > m(j)$, if no such j exists, terminate. Otherwise, repeat the process in step 4 with the following change. In step 4(c)iii, if j' is a firm that was not selected in a previous step, then repeat step 4(c)iii without alteration. However, if j' was selected in a previous step, the above method leads to cycles. To see why, note that Lemma 10 implies any non-zero decrease in p_j may cause demand for j and j' to oscillate. Modify the procedure by selecting all firms j' with equal Gittins indices that were previously selected, and simultaneously lower all corresponding $p_{j'}$s along with p_j in proportions such that $GI_i(j) = GI_i(j')\forall j'$ while also reallocating demand between the set of firms accordingly such that no firm in the set is left with excess capacity.

   This process must end either through all selected firms equating demand and capacity or with some $p_{\tilde{f}} = 0$. In the first event, return to the beginning of step 5, in the latter event, step 4(c)iii can be resumed as normal.

By design, the threshold adjustment process always *increases* the thresholds of various firms. Then, the gross substitutes condition implies that unselected firms never have their demand decrease under the process. Furthermore, step 5 ensures that previously selected firms never face over-demand. Firm j's step only concludes when $D^j = m(j)$. The only unselected firms are those such that $D^j(0,1) \leq m(j)$. Then, the procedure must conclude within a finite number of iterations, at most one per firm. Upon termination, all firms have demand equal to supply. By construction, worker incentives are incorporated through the Gittins index and threshold characterization, therefore the procedure finds an equilibrium, concluding the argument. ■

It is worth emphasizing that the threshold adjustment process converges due to the optimality of forward induction. In particular, had type values been correlated between firms, increasing the threshold of one firm could *decrease* demand at another firm. Alternatively phrased, independence of types is a key component of gross substitutes. Gross substitutes requires that the demand for a firm is weakly increasing in the prices (thresholds) of other firms. Gross substitutes proves to be a weaker sufficient condition in order for firm thresholds to characterize equilibrium. Notably, as shown in Appendix section B.1, a model of gradual learning—where instead of receiving θ_w^j from a match, a worker received a noisy signal of θ_w^j—would also converge to equilibrium under the threshold adjustment process as the gross substitutes condition would still be satisfied.

Proof of Lemma 5

To prove the claim, I first characterize the equilibrium strategy profile in the limit as $\delta \to 1$. I then show that the resulting outcome coincides with the equilibrium outcome in the market with complete information. As $\delta \to 1$, the maximum possible match value must be fully utilized; either the associated firm reaches capacity, or by the associated worker type fully applies to the firm. Otherwise, for sufficiently high δ, a worker of the class with the maximal match value would benefit from applying immediately to the firm, and learning their match value. Then, an extension of the Top-Down algorithm can be used to characterize long-lived workers' strategies.

One key alteration is necessary. Since, workers have incomplete information, rather than specifying the application strategy of a given match value, the algorithm now specifies the application strategy for an entire class. Then, agents no longer behave identically every period. Instead, workers of a given class follow a descending chain of applications, hunting for their top match value. When doing so, the mass of each type within a class must decrease by $1 - \delta$ every step as workers retire along the way. However, as $\delta \to 1$, this decrease in the mass of the overall class collapses to zero, and therefore does not affect the limiting outcome.

To proceed, consider the previous Top-Down algorithm, where workers now follow the strategy for their entire class, stopping searching once they have found their proscribed match value. Notably, when a firm's capacity is not exhausted in a given iteration, workers immediately learn if they would achieve the relevant match value upon applying once. Either they are accepted and learn their actual match value, or they are rejected and learn that they do not have the match value specified in that step of the procedure. If a firm's capacity were to be exhausted for a given iteration, it would be possible for a rejected worker to have access to the associated match value, but simply be unlucky. Then, equilibrium behavior may require that workers of such a class apply multiple times to the same firm. Similar to the complete information setting, one of two cases results, either oversubscription or undersubscription.

Type-convergence holds inductively. Note that there are a finite number of steps, and each step differs by a continuous function of δ. First, the top type converges as $\delta \to 1$. Then, the distance between the outcomes of $\mathcal{M}$ and $\mathcal{M}^I$ is a continuous function of δ. When types are discrete, the rest converge immediately as well. ∎

Proof of Lemma 6

At the first step in which oversubscription occurs in the algorithm of Lemma 1, select the associated type-θ^* and firms j^* and j_2. Type-i^* workers then either apply to j^* or j_2. Those that fail to apply to j^* will never return, while those who apply to j^* are rejected with positive probability. This generates path dependence in equilibrium. ∎

Proof of Proposition 5

To begin, note that congestion implies that the total number of workers hired does not change under information revelation. In particular, market congestion implies that several workers who applied to firm j under $\mathcal{M}$ were rejected. Furthermore, the rejected workers must have weakly lower match values than the workers who are hired. If θ_h was maximally hired at firm j, then the total payoff for θ_h at firm j cannot increase. However, since θ_l must decrease its level of applications to firm j relative to θ_h, θ_l must apply in larger quantities to firms other than j. $|\mathcal{C}| = 1$ requires θ_h and θ_l to share a class. In expectation, their match values at other firms are equivalent. As such, the net effect is a reduction in the proportion of θ_l rejected at j, increasing the congestion at the remaining set of firms. This causes the average payoff of workers at those firms to decrease. This argument proves parts 1 and 2 of the claim. When $m(j) < m(\theta_h)$ this can be the only effect on θ_h type workers, and so their payoff decreases, however type-θ_l workers see a commensurate increase in payoff since they are no longer being rejected from firm j. Therefore, the change in payoffs must be greater for θ_l relative to θ_h. ■

Proof of Lemma 7

Since types θ_h and θ_l are in class c, workers of either type have Gittins indices that are equal across every firm. As a reminder, the optimal strategy every period for a worker was to apply to the firm with the highest Gittins index. In particular, a worker stops searching if her match value at a given firm is above the Gittins index of any other firm. That is, whenever θ_ls stop searching at some firm j, $GI_{\theta_l}(j) \geq GI_{\theta_l}(j') \forall j'$. Since both worker types are in class c, this implies that both types follow the same probability distribution over initial applications. Furthermore, $\theta_h^j \geq \theta_l^j$ by assumption, and so $\theta_h^j \geq GI_{\theta_h}(j')$. Therefore, $s_{\theta_h} \leq s_{\theta_l}$. ■

Proof of Proposition 6

For any market, $\mathcal{M}$, and profile of payment functions $\{\phi_j\}_{j \in \mathcal{F}}$, observe that a new market, $\mathcal{M}_\phi$, can be defined to incorporate payment functions into match values.

Definition. *Let market $\mathcal{M} = (\mathcal{F}, \mathcal{C}, m)$ be given.*

Then, market $\mathcal{M}_\phi = (\mathcal{F}', \mathcal{C}', m')$, where:

1. $\mathcal{F}' = \mathcal{F}$,
2. $\mathcal{C}' = \cup_{c \in \mathcal{C}} \{\theta' = (\theta_w^1 + \phi_1(\theta), \theta_f^1 - \phi_1(\theta), \ldots) | \theta \in c\}$,
3. $m'(\theta') = m(\theta)$.

Then, observe that $\mathcal{M}_\phi$ satisfies all the assumptions of the model. Limited liability implies that match values are always above zero. Furthermore, because wages could only be conditioned on match value, not type, the types in $\mathcal{M}_\phi$ satisfy independence conditional on a worker's class. Then, Proposition 3 shows that aggregate demand in $\mathcal{M}_\phi$ satisfies gross substitutes. ■

B Extensions Satisfy Gross Substitutes

This section shows that the bandit structure ensures that gross substitutes is satisfied in more general settings. To do so, I build on Lemma 4 to extend the model in a natural manner. In each instance, the set of firms remains a simple family of multi-armed bandits, and furthermore, worker's Gittins indices are decreasing in each firms' thresholds. Then, Lemma 4 implies that gross substitutes is satisfied, and the threshold adjustment process of Theorem 1 can be used to characterize an equilibrium.

B.1 Gradual Learning

Throughout the paper so far, it has been assumed that workers learn immediately—a type-θ worker that is hired by firm j learns θ_w^j. In many scenarios, learning requires time, or is noisy. Here, I show that in markets where workers receive a noisy signal of their match value, aggregate demand still satisfies gross substitutes.

Noise is parametrized through a normal distribution. When a type-θ worker, i, is hired by firm j in period t, i observes her utility from the match, which is given by $\theta_w^j + \epsilon_i(t)$. $\epsilon_i(t)$ is drawn iid each period from a normal distribution, $\epsilon_i(t) \sim N(0, \xi)$, where $\xi > 0$.

Lemma 11. *Aggregate demand satisfies gross substitutes when learning is gradual.*

Proof of Lemma 11

To begin, I note that workers face a simple family of multi-armed bandit processes. For any firm j, associated thresholds, (v_j, p_j), and type-θ; the expected payoff for type-θ from applying to j is equivalent to type-θ's expected payoff from applying to j when learning was instant.

Then, it remains to be shown that $GI_\theta(j, h_t^\theta)$ is decreasing in (v_j, p_j). A simple interchange argument proves the point. Suppose the optimal stopping solution to $GI_\theta(j, h_t^\theta)$ yielded a larger value for some $(v_j', p_j') > (v_j, p_j)$. Then, utilize the stopping solution for (v_j', p_j') in place of the original stopping solution for (v_j, p_j), with one key difference. Observe that the value from the new solution, under (v_j, p_j) can be decomposed into two terms, one corresponding to matches would have been received under (v_j', p_j'), and a second corresponding to the value for the additional matches due to the difference between (v_j, p_j) and (v_j', p_j'). Then, possibly through garbling the original stopping solution, a new stopping solution can be characterized that mimics the original stopping solution for matches above (v_j', p_j') on average, while treating matches between (v'_j, p'_j) and (v_j, p_j) as if they had led to rejection. This new stopping solution's value can then be decomposed into two terms, one of which is equal to the original stopping solution's value, and the second which includes the benefit from matching for a single period, and is therefore positive.

Last, Lemma 4 implies that individual demand satisfies gross substitutes, and therefore aggregate demand satisfies gross substitutes. ■

B.2 Learning Through Interviews

Similar to the previous result, when workers learn at the interview stage, the gross substitutes condition holds. Formally, when a type-θ worker applies to firm j, the worker learns θ_w^j, regardless of whether she is hired.

Lemma 12. *Aggregate demand satisfies gross substitutes when workers learn through interviewing.*

Proof of Lemma 12

Learning through interviewing simplifies the previous constructions. Previously, when a type-θ worker applied to a firm with threshold (v_j, p_j), if $\theta_w^j > v_j$ she learned as much. If instead $\theta_w^j \leq v_j$, then she may have been unable to tell whether $\theta_w^j = v_j$ and she was unlucky or whether $\theta_w^j < v_j$. Now, she faces a simple family of multi-armed bandit processes where the reward from each bandit j is the realization of Ψ_θ^j in expectation. Then, by construction, $GI_\theta(j, h_t^\theta)$ is decreasing in (v_j, p_j) and therefore Lemma 4 implies that individual demand satisfies gross substitutes. It follows that aggregate demand satisfies gross substitutes as well. ■

B.3 Heterogeneous Discounting

In practice, different classes of workers may not only have different priors regarding their type, but they may also retire from the market at different rates. For instance, if different classes also correspond to different ages of workers, those classes may retire at individual rates. Suppose each class c retires from the market with probability $\delta_c \in [0, 1)$.

Lemma 13. *Aggregate demand satisfies gross substitutes when different classes have distinct discount factors.*

Proof of Lemma 13

For each class of workers, the set of firms acts as a multi-armed bandit problem, albeit with differing values of δ_c. Furthermore, the monotonicity of the Gittins index in rewards implies that $GI_\theta(j, h_t^i)$ is decreasing in (v_j, p_j). As such, Theorem 2 implies that each type's demand satisfies gross substitutes. By the previous arguments it follows that aggregate demand satisfies gross substitutes. ■